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The non-existence of a regular maximal connected expansion of the reals

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ABSTRACT

We prove that there is no regular maximal connected expansion of the Euclidean topology ε on the set $\mathbb{R}$ of real numbers. For this, we first consider a Hausdorff connected submaximal space $(\mathbb{R}, \tau)$ with $\varepsilon \subseteq \tau$ and then, with the aid of the filter of the τ-dense sets, we define two specific expansions σ, τ^* of ε, such that $\tau^* \subseteq \sigma, \tau^* \subseteq \tau$ and $(\mathbb{R}, \sigma)$ is submaximal. We prove that if $(\mathbb{R}, \tau)$ is in addition nearly maximal connected, then $\sigma = \tau^*$. Finally we prove that $(\mathbb{R}, \tau)$ cannot be regular.

1. Introduction

In [3], El'kin posed the question whether there exists a regular maximal connected space whose topology is finer than the Euclidean topology ε on the set $\mathbb{R}$ of real numbers. The same problem is mentioned by Pavlov ([12], Question 5) where many similar problems are presented in detail. Maximal connected spaces are studied in [2–11] and [13]. We note that the spaces constructed in [5] and [13] are Hausdorff maximal connected whose topologies are finer than ε.

Definition 1.1. A Hausdorff space (X, τ) is called:

(1) *Maximal connected*, if every topology strictly finer than τ is not connected.

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(2) *Nearly maximal connected* ([2]), if (X, τ) is connected and for all regular open sets V and for all x in the boundary of V, there exists a neighborhood U_1 of x and an open non-empty set U_2 such that $U_1 \cap V \cap U_2 = \emptyset$ and $U_2 \cup \{x\}$ is closed.
(3) *Submaximal*, if every dense subset is open.

Let $(\mathbb{R}, \tau)$ with $\varepsilon \subseteq \tau$ and let O be a τ-open neighborhood of $x \in \mathbb{R}$. We set $O^- = (-\infty, x) \cap O$ and $O^+ = (x, +\infty) \cap O$. As it is proved in [2], a connected space is maximal connected if and only if it is submaximal and nearly maximal connected. This result, in case of maximal connected space $(\mathbb{R}, \tau)$ with $\varepsilon \subseteq \tau$, implies that if U is a τ-open set such that $U \subseteq (x, +\infty)$ (resp. $U \subseteq (-\infty, x)$) and x is a τ-accumulation point of U, then $\{x\} \cup U$ is a τ-open neighborhood of x in the subspace $[x, +\infty)$ (resp. $(-\infty, x]$). It also implies that every connected set in ε remains connected in τ ([6], [7]). Obviously (by [2]) a regular maximal connected space is Hausdorff maximal connected.

In order to prove that there does not exist a regular maximal connected space whose topology is finer than ε, we first consider a Hausdorff connected submaximal space $(\mathbb{R}, \tau)$ with $\varepsilon \subseteq \tau$. Using the filter of τ-dense sets, we define two specific expansions σ, τ^* of ε, such that $\tau^* \subseteq \sigma, \tau^* \subseteq \tau$ and $(\mathbb{R}, \sigma)$ is submaximal. Then, if in addition $(\mathbb{R}, \tau)$ is nearly maximal connected (hence $(\mathbb{R}, \tau)$ is Hausdorff maximal connected), we prove that $\sigma = \tau^*$. Finally, if $(\mathbb{R}, \tau)$ is in addition regular, we prove that this leads to a contradiction.

2. The spaces $(\mathbb{R}, \sigma)$ and $(\mathbb{R}, \tau^*)$

Let $(\mathbb{R}, \tau)$ be a Hausdorff connected submaximal space with $\varepsilon \subseteq \tau$. The set $\mathcal{L}$ of τ-dense sets is a maximal filter on this set. Obviously $\mathcal{L}$ is a filter on the set of ε-dense sets. We expand the filter $\mathcal{L}$ to a maximal filter $\mathcal{D}$ on the set of ε-dense sets and we set $\sigma =< \varepsilon \cup \mathcal{D} >$.

Lemma 2.1. *The space* $(\mathbb{R}, \sigma)$ *has the following properties:*

(1) $\varepsilon \subseteq \sigma$.
(2) *It is Hausdorff connected submaximal.*
(3) *Every* τ*-dense set is* σ*-dense.*
(4) *Every* τ*-closed and* τ*-discrete set is* σ*-closed and* σ*-discrete.*
(5) *If* D *is* σ*-dense, then the set* $Int_\tau D$ *is also* σ*-dense.*

Proof. (1) It is obvious.
(2) The topology σ is obtained by adjoining a maximal filter $\mathcal{D}$ of ε-dense sets, hence $(\mathbb{R}, \sigma)$ is Hausdorff connected submaximal ([1], [5]).
(3) It follows from the fact that $\mathcal{L} \subseteq \mathcal{D}$.
(4) Let M be a τ-closed and τ-discrete set. Since $(\mathbb{R}, \tau)$ is connected, no point of M or of $\mathbb{R} \setminus M$ is τ-isolated. Hence the set $\mathbb{R} \setminus M$ is τ-open and τ-dense. By the previous property (3), the set $\mathbb{R} \setminus M$ is σ-dense and σ-open because $(\mathbb{R}, \sigma)$ is submaximal. Therefore M is σ-closed and σ-discrete.
(5) First we observe that if $Int_\tau D = \emptyset$ then D is τ-closed and τ-discrete, hence by the previous property (4) it is also σ-closed and σ-discrete. Since the set $\mathbb{R} \setminus D$ is also σ-closed and σ-discrete it follows that the space $(\mathbb{R}, \sigma)$ is discrete, which is impossible because it is connected. Hence $Int_\tau D \neq \emptyset$. The set $D \setminus Int_\tau D$ is τ-closed and τ-discrete. Again, by property (4), it is also σ-closed and σ-discrete. Hence the set $(D \setminus Int_\tau D) \cup (\mathbb{R} \setminus D)$ is σ-closed and σ-discrete. Therefore the set $Int_\tau D$ is σ-dense. □

Although the topologies τ and σ are not comparable, we can define another topology τ^* which is weaker than both of them. We observe at first that the set $Int_\tau \mathcal{D} = \{Int_\tau D \mid D \text{ is } \sigma\text{-dense }\}$ is a base for a topology consisting of sets σ-dense, ε-dense and τ-open. Obviously $\mathcal{L} \subseteq Int_\tau \mathcal{D}$. We set $\tau^* =< \varepsilon \cup Int_\tau \mathcal{D} >$.