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On conjugates and adjoint descent

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ABSTRACT

In this note we present an ∞-categorical framework for descent along adjunctions and a general formula for classifying conjugates up to equivalence, which unifies several known formulae from different fields.

1. Introduction

The notion of "conjugate objects" or "objects of the same genus" arises in many fields in mathematics: in commutative algebra as objects that become isomorphic after a field extension ([8]), in homotopy theory as spaces that have equivalent Postnikov truncations ([10]) and in group theory as nilpotent groups that have isomorphic localizations ([2]). Often, one also has a formula computing the set of conjugates of a given object. In the three examples mentioned above, those sets are given in terms of Galois cohomology, $\lim^1$ of a tower of groups and a double coset formula respectively.

The goal of this paper is twofold:

A. To unify and generalize the examples above by giving an abstract ∞-categorical definition of conjugates (Definition 1.2) and a general formula for classifying them (Theorem A).
B. To prove a descent result which facilitates the construction of the above ∞-categorical framework in many cases of interest (Theorem B and its dual, Corollary 1.3).

In what follows we always work in the setting of ∞-categories[1] using heavily the results and terminology of [5]. In particular, $\mathbf{Cat}_\infty$ is the ∞-category of ∞-categories (see [5, Definition 3.0.0.1]) and we use the

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[1] Also known as 'quasi-categories' or 'weak Kan complexes'.

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symbols $\varprojlim$ and $\varinjlim$ for the (∞-categorical) limit and colimit of a functor between ∞-categories. We also denote by $\mathcal{C}^{\simeq}$ the maximal ∞-subgroupoid of an ∞-category $\mathcal{C}$ and we abuse notation by identifying ordinary categories with their nerves viewed as ∞-categories.

For a general definition of conjugate objects, we first need to fix some notation. Let I be a simplicial set. An I-diagram of ∞-categories is a map $I \to \mathbf{Cat}_\infty$, which we denote by $\mathcal{D}_\bullet$ (where $\mathcal{D}_a$ is the image of a vertex $a \in I$). A cone on $\mathcal{D}_\bullet$ is an extension of the map $I \to \mathbf{Cat}_\infty$ to the cone $I^{\triangleleft}$. We denote such a cone by $\mathcal{C} \to \mathcal{D}_\bullet$, where $\mathcal{C}$ is the image of the cone point. In this situation, by the universal property of the limit, we get a canonical functor $\mathcal{C} \to \varprojlim(\mathcal{D}_\bullet)$. We call this the *comparison functor* of the cone. Now assume that we are in the following setting:

Setting 1.1. Let I be a simplicial set, $\mathcal{D}_\bullet$ an I-diagram of ∞-categories, and $\mathcal{C} \to \mathcal{D}_\bullet$ a cone. Denote by $F_a \colon \mathcal{C} \to \mathcal{D}_a$ the functor corresponding to the edge from the cone point to $a \in I$ and by $F \colon \mathcal{C} \to \varprojlim(\mathcal{D}_\bullet)$ the comparison functor (see Fig. 1).

$$
\begin{tikzcd}
 & & \mathcal{D}_a \arrow[dr] \arrow[dd] & \\
\mathcal{C} \arrow[r, "F"] \arrow[urr, bend left, "F_a"] \arrow[rrr, bend left, "F_b"] \arrow[drr, bend right, "F_c"'] & \varprojlim(\mathcal{D}_\bullet) \arrow[ur] \arrow[rr, bend left] \arrow[dr] & & \mathcal{D}_b \arrow[dl] \\
 & & \mathcal{D}_c . &
\end{tikzcd}
$$

Fig. 1. A cone on an I-diagram of ∞-categories and the comparison functor.

Given two objects $x, y \in \mathcal{C}$ one can try to distinguish between them by comparing $F_a(x)$ and $F_a(y)$ in $\mathcal{D}_a$. If $F_a(x)$ and $F_a(y)$ fail to be equivalent for some $a \in I$, then clearly x and y can not be equivalent in $\mathcal{C}$. Two objects x and y are called *conjugate* if they can't be distinguished in this way. More formally,

Definition 1.2. In the Setting 1.1, two objects x and y in $\mathcal{C}$ will be called *conjugate* if there exist (not necessarily compatible) equivalences $F_a(x) \simeq F_a(y)$ for every index a in I. Let $\mathrm{Conj}(x) \subseteq \mathcal{C}^{\simeq}$ denote the full ∞-subgroupoid of conjugates of x.

In addition, for every object x of an ∞-category $\mathcal{C}$, we denote by $\mathrm{BAut}(x) \subseteq \mathcal{C}^{\simeq}$ the full sub ∞-groupoid spanned by the single object x (i.e. the maximal Kan subcomplex of the simplicial set $\mathcal{C}$ supported on a single vertex x).

We explain the terminology as follows. By identifying ∞-groupoids with spaces, we can think of $\mathrm{BAut}(x)$ as a connected space pointed by x. The loop space $\Omega\,\mathrm{BAut}(x)$ is homotopy equivalent to the space $\mathrm{Aut}(x) \subseteq \mathrm{Map}_{\mathcal{C}}(x, x)$ of self-equivalences of x with the loop space structure corresponding to composition of maps. Thus, $\mathrm{BAut}(x)$ is the classifying space of $\mathrm{Aut}(x)$.[2]

With these definitions, our main results are:

Theorem A *(Conjugates formula). In the Setting 1.1, if the comparison functor $F \colon \mathcal{C} \to \varprojlim(\mathcal{D}_\bullet)$ is an equivalence, then it induces an equivalence of ∞-groupoids:*[3]

[2] In the equivalent context of topological categories, this is precisely [5, Remark 1.2.5.2].

[3] Note that the ∞-limit of ∞-groupoids corresponds to the homotopy limit of spaces under the identification of the ∞-categories of ∞-groupoids and spaces.