

On conjugates and adjoint descent



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ABSTRACT

In this note we present an ∞ -categorical framework for descent along adjunctions and a general formula for classifying conjugates up to equivalence, which unifies several known formulae from different fields.

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1. Introduction

The notion of “conjugate objects” or “objects of the same genus” arises in many fields in mathematics: in commutative algebra as objects that become isomorphic after a field extension ([8]), in homotopy theory as spaces that have equivalent Postnikov truncations ([10]) and in group theory as nilpotent groups that have isomorphic localizations ([2]). Often, one also has a formula computing the set of conjugates of a given object. In the three examples mentioned above, those sets are given in terms of Galois cohomology, $\lim^1$ of a tower of groups and a double coset formula respectively.

The goal of this paper is twofold:

- A. To unify and generalize the examples above by giving an abstract ∞ -categorical definition of conjugates (Definition 1.2) and a general formula for classifying them (Theorem A).
- B. To prove a descent result which facilitates the construction of the above ∞ -categorical framework in many cases of interest (Theorem B and its dual, Corollary 1.3).

In what follows we always work in the setting of ∞ -categories¹ using heavily the results and terminology of [5]. In particular, $\mathbf{Cat}_\infty$ is the ∞ -category of ∞ -categories (see [5, Definition 3.0.0.1]) and we use the

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¹ Also known as ‘quasi-categories’ or ‘weak Kan complexes’.

symbols $\varprojlim$ and $\varinjlim$ for the (∞ -categorical) limit and colimit of a functor between ∞ -categories. We also denote by $\mathcal{C}^\simeq$ the maximal ∞ -subgroupoid of an ∞ -category $\mathcal{C}$ and we abuse notation by identifying ordinary categories with their nerves viewed as ∞ -categories.

For a general definition of conjugate objects, we first need to fix some notation. Let I be a simplicial set. An I -diagram of ∞ -categories is a map $I \rightarrow \mathbf{Cat}_\infty$, which we denote by $\mathcal{D}_\bullet$ (where $\mathcal{D}_a$ is the image of a vertex $a \in I$). A cone on $\mathcal{D}_\bullet$ is an extension of the map $I \rightarrow \mathbf{Cat}_\infty$ to the cone $I^\triangleleft$. We denote such a cone by $\mathcal{C} \rightarrow \mathcal{D}_\bullet$, where $\mathcal{C}$ is the image of the cone point. In this situation, by the universal property of the limit, we get a canonical functor $\mathcal{C} \rightarrow \varprojlim(\mathcal{D}_\bullet)$. We call this the *comparison functor* of the cone. Now assume that we are in the following setting:

Setting 1.1. Let I be a simplicial set, $\mathcal{D}_\bullet$ an I -diagram of ∞ -categories, and $\mathcal{C} \rightarrow \mathcal{D}_\bullet$ a cone. Denote by $F_a: \mathcal{C} \rightarrow \mathcal{D}_a$ the functor corresponding to the edge from the cone point to $a \in I$ and by $F: \mathcal{C} \rightarrow \varprojlim(\mathcal{D}_\bullet)$ the comparison functor (see Fig. 1).

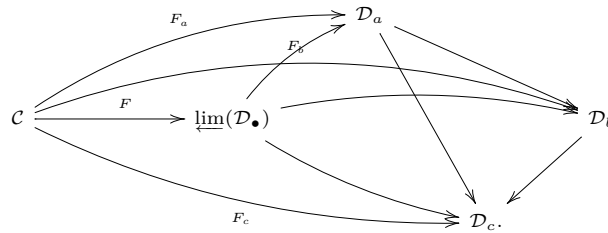


Fig. 1. A cone on an I -diagram of ∞ -categories and the comparison functor.

Given two objects $x, y \in \mathcal{C}$ one can try to distinguish between them by comparing $F_a(x)$ and $F_a(y)$ in $\mathcal{D}_a$. If $F_a(x)$ and $F_a(y)$ fail to be equivalent for some $a \in I$, then clearly x and y can not be equivalent in $\mathcal{C}$. Two objects x and y are called *conjugate* if they can't be distinguished in this way. More formally,

Definition 1.2. In the Setting 1.1, two objects x and y in $\mathcal{C}$ will be called *conjugate* if there exist (not necessarily compatible) equivalences $F_a(x) \simeq F_a(y)$ for every index a in I . Let $\text{Conj}(x) \subseteq \mathcal{C}^\simeq$ denote the full ∞ -subgroupoid of conjugates of x .

In addition, for every object x of an ∞ -category $\mathcal{C}$, we denote by $\text{BAut}(x) \subseteq \mathcal{C}^\simeq$ the full sub ∞ -groupoid spanned by the single object x (i.e. the maximal Kan subcomplex of the simplicial set $\mathcal{C}$ supported on a single vertex x).

We explain the terminology as follows. By identifying ∞ -groupoids with spaces, we can think of $\text{BAut}(x)$ as a connected space pointed by x . The loop space $\Omega \text{BAut}(x)$ is homotopy equivalent to the space $\text{Aut}(x) \subseteq \text{Map}_{\mathcal{C}}(x, x)$ of self-equivalences of x with the loop space structure corresponding to composition of maps. Thus, $\text{BAut}(x)$ is the classifying space of $\text{Aut}(x)$.²

With these definitions, our main results are:

Theorem A (Conjugates formula). In the Setting 1.1, if the comparison functor $F: \mathcal{C} \rightarrow \varprojlim(\mathcal{D}_\bullet)$ is an equivalence, then it induces an equivalence of ∞ -groupoids:³

² In the equivalent context of topological categories, this is precisely [5, Remark 1.2.5.2].

³ Note that the ∞ -limit of ∞ -groupoids corresponds to the homotopy limit of spaces under the identification of the ∞ -categories of ∞ -groupoids and spaces.

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