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snf-Countability and csf-countability in $F_4(X)$ ☆

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A B S T R A C T

Let $F(X)$ be the free topological group on a Tychonoff space X, and $F_n(X)$ the subspace of $F(X)$ consisting of all words of reduced length at most n for each $n \in \mathbb{N}$. In this paper conditions under which the subspace $F_4(X)$ of the free topological group $F(X)$ on a generalized metric space X contains no closed copy of S_ω are obtained and used to discuss countability axioms in free topological groups. It is proved that for a k-semistratifiable k-space X the subspace $F_4(X)$ is snf-countable if and only if X is compact or discrete; for a normal k- and $\aleph$-space X $F_4(X)$ is csf-countable if and only if X is an $\aleph_0$-space or discrete; and for a k^*-metrizable space X $F_5(X)$ is a k-space and $F_4(X)$ is csf-countable if and only if X is a k_ω-space or discrete. Some results of K. Yamada, and F. Lin, C. Liu and J. Cao are improved.

1. Introduction

The symbols $F(X)$ and $A(X)$ denote respectively the *free topological group* and the *free Abelian topological group* on a Tychonoff space X in the sense of Markov [25]. Free topological groups have become a powerful tool of investigation in the theory of topological groups that serve as a source of various examples and as

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an instrument for proving new theorems [3]. We use $G(X)$ to denote either $F(X)$ or $A(X)$. It is a natural question whether there is a topological property P of a space X which characterizes a topological property Q of $G(X)$. For example, the question of on what space X the free topological group $G(X)$ is a k-space has been studied by several topologists. It is a classic result that a space X is a k_ω-space if and only if so is the group $G(X)$ [24]. Arhangel'skiĭ, Okunev and Pestov [2] proved that the topological group $F(X)$ on a metrizable space X is a k-space if and only if X is locally compact and separable or discrete.

Let $\mathbb{N}$ be the set of all positive integers. In what follows, for each $n \in \mathbb{N}$ $F_n(X)$ and $A_n(X)$ stand for the subset of $F(X)$ and $A(X)$ formed by all words whose length is less than or equal to n, respectively. Thus, any statement about $G_n(X)$ applies to both $F_n(X)$ and $A_n(X)$. It is well known that if a space X is not discrete, then neither $A(X)$ is first-countable nor $F(X)$ is Fréchet–Urysohn (see [3, Theorem 7.1.20] and [13, Corollary 4.17]). However, $F_n(X)$ and $A_n(X)$ have a chance to be first-countable or Fréchet–Urysohn for a non-discrete space X. These facts motivate researchers to investigate the countability axioms of free topological groups in the following two directions [15]: one is to study some weak forms of countability axioms in $F(X)$ or $A(X)$ over certain classes of spaces X; another is to study some weak forms of countability axioms in $F_n(X)$ or $A_n(X)$ over certain classes of spaces X.

The set of all non-isolated points of a space X is denoted by $NI(X)$ in this paper. Let X be a Tychonoff space. Denote by X^{-1} a copy of a space X and by e the identity of the free group $G(X)$. The mapping $i_n : (X \oplus \{e\} \oplus X^{-1})^n \to G_n(X)$ is defined by $i_n((x_1, x_2, \cdots, x_n)) = x_1 x_2 \cdots x_n$ for each $n \in \mathbb{N}$.

K. Yamada [33–35] made a systematic and outstanding work in the two research directions over metrizable spaces. The following results were obtained.

Theorem 1.1. *([33,34]) The following are equivalent for a metrizable space X:*

(1) *$A_n(X)$ is metrizable for each $n \in \mathbb{N}$;*
(2) *$A_3(X)$ is Fréchet–Urysohn;*
(3) *$F_3(X)$ is metrizable;*
(4) *$G_2(X)$ is first-countable;*
(5) *i_2 is a closed mapping;*
(6) *$NI(X)$ is compact.*

Theorem 1.2. *([33,34]) The following are equivalent for a metrizable space X:*

(1) *$F_n(X)$ is metrizable for each $n \in \mathbb{N}$;*
(2) *$F_5(X)$ is Fréchet–Urysohn;*
(3) *$F_4(X)$ is first-countable;*
(4) *i_n is a closed mapping for each $n \in \mathbb{N}$;*
(5) *i_4 is a closed mapping;*
(6) *X is compact or discrete.*

Theorem 1.3. *([35]) The following are equivalent for a metrizable space X:*

(1) *$F(X)$ is a k-space;*
(2) *$F_n(X)$ is a k-space for each $n \in \mathbb{N}$;*
(3) *X is locally compact separable or discrete.*

These results are beautiful, but slightly incomplete, which leaves a space for further research. For example, when, in terms of the space X, is the subspace $F_4(X)$ or $F_3(X)$ Fréchet–Urysohn? Is the mapping i_3 closed? Is $F_n(X)$ a k-space for some $k \in \mathbb{N}$? Recently, F. Lin, C. Liu et al. [12,13,15] attempted to extend Yamada's