Topology and its Applications 232 (2017) 183–188

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Homogeneity degree of cones

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ARTICLE INFO

Article history:
Received 15 January 2017
Received in revised form 11 October 2017
Accepted 12 October 2017
Available online 14 October 2017

MSC:
54B15
54F15
54F50

Keywords:
Locally connected curve
Dendrite
Cone
Homogeneity degree
Orbit

ABSTRACT

The homogeneity degree of a space X is the number of orbits for the action of the autohomeomorphisms group of X. We determine the homogeneity degree of the cone over a locally connected curve X not being a local dendrite in terms of that of X. Using the result of Pellicer-Covarrubias and Santiago-Santos, it gives us a formula for the homogeneity degree of the cone over any locally connected curve X.

1. Introduction

Let X be a topological space. The *cone* of X is the quotient space

$$\mathrm{Cone}(X) = X \times I \diagup X \times \{1\}.$$

Let $\mathcal{H}(X)$ denote the group of autohomeomorphisms of X. Given $x \in X$, we denote by $\mathcal{O}_X(x)$ the orbit of x under the action of $\mathcal{H}(X)$ on X:

$$\mathcal{O}_X(x) = \{h(x) : \ h \in \mathcal{H}(X)\}.$$

We say that X is $\frac{1}{n}$-*homogeneous* provided that X has exactly n orbits. The *homogeneity degree* of X is n (in symbols $d_H(X) = n$) if X is $\frac{1}{n}$-homogeneous.

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https://doi.org/10.1016/j.topol.2017.10.010
0166-8641/© 2017 Published by Elsevier B.V.

The homogeneity degree has been studied recently in many papers, (see e.g. [3], [4], and [7–10]).

By $\alpha(X)$ we denote the points of X having a neighbourhood homeomorphic to the Euclidean space $\mathbb{E}^n$, by $\beta(X)$ – the points of $X \setminus \alpha(X)$ having a neighbourhood homeomorphic to the half Euclidean space.

The main theorem of this note reads as follows:

Theorem 1. *Let us assume that X is a locally connected curve not being a local dendrite. If $d_H(X)$ is finite and k is the number of orbits of X contained in $\beta(X)$, then $d_H(\mathrm{Cone}(X)) = 2d_H(X) + 1 - 2k$. If $d_H(X)$ is infinite then $d_H(X) = d_H(\mathrm{Cone}(X))$.*

For every $n \in \mathbb{N}$, let T_n and Θ_n denote the cone and suspension over an n-point set, respectively. A *hairy point* F_ω is the union of arcs A_i, $i \in \mathbb{N}$, such that $\mathrm{diam} A_i \to 0$ and $A_i \cap A_j = \{a\}$, for all $i \neq j \in \mathbb{N}$.

In [10], Pellicer-Covarrubias and Santiago-Santos determined the homogeneity degree of the cones over local dendrites in terms of that of X. More precisely, they proved the following result:

Theorem 2. *Let X be a local dendrite.*

(1) *If X is a simple closed curve or $X \in \{F_\omega\} \cup \{T_n : n \in \mathbb{N} \setminus \{1, 2\}\}$ then $d_H(\mathrm{Cone}(X)) = d_H(X) + 1$.*
(2) *If $X \in \{\Theta_n : n \in \mathbb{N} \setminus \{1, 2\}\}$ then $d_H(\mathrm{Cone}(X)) = d_H(X) + 2 = 4$.*
(3) *If $d_H(X)$ is infinite or X is an arc, then $d_H(\mathrm{Cone}(X)) = d_H(X)$.*
(4) *In all other cases, $d_H(\mathrm{Cone}(X)) = 2d_H(X) + 1 - 2k$, where k is the number of orbits of X contained in $\beta(X)$.*

By Theorems 1 and 2, we obtain a formula for the homogeneity degree of the cone over X in terms of that of X, for X being any locally connected curve.

The proofs of the main lemmas in this paper involve techniques and ideas developed in [1] and employ the notation of isotopic components. This part of our work is contained in Section 3.

Section 4 contains the proof of Theorem 1.

Using similar tools the author gives in [6] a formula for the homogeneity degree of the suspension over a locally connected curve X not being a local dendrite in terms of that of X.

2. Notation and tools

Our terminology follows [2]. All spaces are assumed to be metric. A curve is a 1-dimensional continuum. A dendrite is a curve being AR.

In [5] (see Lemma 5.1 on the page 43), the author proved the following result.

Lemma 1. *If C and C' are locally connected curves not being local dendrites and $h\colon \mathrm{Cone}(C) \to \mathrm{Cone}(C')$ is a homeomorphism then h maps the vertex of $\mathrm{Cone}(C)$ onto the vertex of $\mathrm{Cone}(C')$.*

Remark 1. ([10], Lemma 3.0.6) Let X be a topological space. If A is contained in an orbit of X, then $A \times (0, 1)$ and $A \times \{0\}$ are contained in orbits of $\mathrm{Cone}(X)$.

Using the remark above we obtain:

Lemma 2. *Let X be a locally connected curve not being a local dendrite and let $\mathcal{O}_{\mathrm{Cone}(X)}(x, t)$ be an orbit of $\mathrm{Cone}(X)$ different from that of the vertex. Then*

$$\mathcal{O}_X(x) \times \{0\} \subseteq \mathcal{O}_{\mathrm{Cone}(X)}(x, t), \quad \text{for} \quad t = 0,$$