Topology and its Applications 232 (2017) 189–213

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The diffeology of Milnor's classifying space

Jean-Pierre Magnot [a,b], Jordan Watts [c,*]

[a] LAREMA, Université d'Angers, 2 Bd Lavoisier, 49045 Angers cedex 1, France
[b] Lycée Jeanne d'Arc, 40 avenue de Grande Bretagne, 63000 Clermont-Ferrand, France
[c] Department of Mathematics, University of Colorado Boulder, Campus Box 395, Boulder, CO 80309, USA

ARTICLE INFO

Article history:
Received 21 June 2016
Received in revised form 10 October 2017
Accepted 11 October 2017
Available online 14 October 2017

MSC:
primary 53C05, 57R55
secondary 58B05, 58B10

Keywords:
Diffeology
Classifying space
Universal bundle
Diffeological group
Lie group

ABSTRACT

We define a diffeology on the Milnor classifying space of a diffeological group G, constructed in a similar fashion to the topological version using an infinite join. Besides obtaining the expected classification theorem for smooth principal bundles, we prove the existence of a diffeological connection on any principal bundle (with mild conditions on the bundles and groups), and apply the theory to some examples, including some infinite-dimensional groups, as well as irrational tori.

1. Introduction

Let G be a topological group with a reasonable topology (Hausdorff, paracompact, and second-countable, say). Milnor [19] constructed a universal topological bundle $EG \to BG$ with structure group G satisfying:

- for any principal G-bundle $E \to X$ over a space (again, assume Hausdorff, paracompact, and second-countable) there is a continuous "classifying map" $F\colon X \to BG$ for which E is G-equivariantly homeomorphic to the pullback bundle F^*EG,
- any continuous map $F\colon X \to BG$ induces a bundle F^*EG with F a classifying map, and
- any two principal G-bundles are isomorphic if and only if their classifying maps are homotopic.

* Corresponding author.
E-mail addresses: jean-pierr.magnot@ac-clermont.fr (J.-P. Magnot), jordan.watts@colorado.edu (J. Watts).

https://doi.org/10.1016/j.topol.2017.10.011

To push this classification into the realm of Lie groups and smoothness, one would need smooth structures on the spaces EG and BG. Many approaches exist, which extend to infinite-dimensional groups (see, for example, [20], [9, Theorem 44.24]). The main point here is that EG and BG are not typically manifolds, and so a more general smooth structure is required.

In this paper, we take the approach of diffeology. The language is quite friendly, allowing one to differentiate and apply other analytical tools with ease to infinite-dimensional groups such as diffeomorphism groups, including those of non-compact manifolds, as well as projective limits of groups, including some groups which appear naturally in the ILB setting of Omori [22], and may not exhibit atlases. Moreover, we can include in this paper interesting groups which are not typically considered as topological groups. For example, irrational tori (see Example 5.3) have trivial topologies (and hence have no atlas) and only constant smooth functions; however, they have rich diffeologies, which hence are ideal structures for studying the groups. Irrational tori appear in important applications, such as pre-quantum bundles on a manifold associated to non-integral closed 2-forms (see Subsection 5.6). Another benefit of using diffeology is that we can directly use the language to construct connection 1-forms on EG. Our main source for the preliminaries on diffeology is the book by Iglesias-Zemmour [8].

Our main results include Theorem 3.6, which states that there is a natural bijection between isomorphism classes of so-called D-numerable principal G-bundles over a Hausdorff, second-countable, smoothly paracompact diffeological space X, and smooth homotopy classes of maps from X to BG. This holds for any diffeological group G. As well, we prove Theorem 4.3, which states that if G is a regular diffeological Lie group, and X is Hausdorff and smoothly paracompact, then any D-numerable principal G-bundle admits a connection; in particular, such a bundle admits horizontal lifts of smooth curves. These theorems provide a method for constructing classifying spaces different, for example, to what Kriegl and Michor do in [9, Theorem 44.24] with $G = \mathrm{Diff}(M)$ for M a compact smooth manifold, where they show that the space of embeddings of M into ℓ^2 yields a classifying space for G.

Our framework is applied to a number of situations. We show that EG is contractible (Proposition 5.1), which allows us to study the homotopy of BG (Proposition 5.2). We also study smooth homotopies between groups, and how these are reflected in classifying spaces and principal bundles (Subsection 5.2). We transfer the theory to general diffeological fibre bundles via their associated principal G-bundles (Corollary 5.9) and discuss horizontal lifts in this context (Proposition 5.11). We also transfer the theory to diffeological limits of groups, with an application to certain ILB principal bundles (Proposition 5.18). We show that a short exact sequence of diffeological groups induces a long exact sequence of diffeological homotopy groups of classifying spaces (Proposition 5.19), and apply this to a short exact sequence of pseudo-differential operators and Fourier integral operators (Example 5.21). Finally, we apply this theory to irrational torus bundles, which are of interest in geometric quantisation [26], [7], [8, Articles 8.40–8.42] and the integration of certain Lie algebroids [5].

This paper is organised as follows. Section 2 reviews necessary prerequisites on diffeological groups (including diffeological Lie groups and regular groups), internal tangent bundles, and diffeological fibre bundles. In Section 3 we construct the diffeological version of the Milnor classifying space, $EG \to BG$, and prove Theorem 3.6. In Section 4, we introduce the theory of connections from the diffeological point-of-view, and prove Theorem 4.3. Finally, in Section 5, we have our applications.

A few open questions are inspired. The conditions on the D-topology (in particular, Hausdorff and second-countable conditions, and sometimes smooth paracompactness as well) seem out-of-place in the general theory of diffeology. Even though these conditions are satisfied in our examples, in some sense, topological conditions and arguments should be *replaced* with diffeological conditions and arguments. This leads to the first question.

Question 1. Under what conditions is the D-topology of a diffeological space Hausdorff, second-countable, and smoothly paracompact? Can one weaken these conditions?