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Notes on countable tightness of the subspaces of free (Abelian) topological groups

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ABSTRACT

Given a Tychonoff space X, let $F(X)$ and $A(X)$ be respectively the free topological group and the free Abelian topological group over X in the sense of Markov. For every $n \in \mathbb{N}$, let $F_n(X)$ (resp. $A_n(X)$) denote the subspace of $F(X)$ (resp. $A(X)$) that consists of words of reduced length at most n with respect to the free basis X. In this paper, we mainly discuss the subspaces $F_n(X)$ and $A_n(X)$ with countable tightness for a Lašnev space X, and prove that:
(1) Assume $\mathfrak{b} = \omega_1$. For a non-metrizable Lašnev space X, the tightness of $F_5(X)$ is countable if and only if the tightness of $F(X)$ is countable;
(2) Let X be the closed image of a locally separable metrizable space. Then the tightness of $A_4(X)$ is countable if and only if the tightness of $A(X)$ is countable.

1. Introduction

Throughout this paper, all topological spaces are assumed to be at least Tychonoff, and all mappings are continuous and onto unless explicitly otherwise stated. Free (Abelian) groups over metric spaces have been widely discussed. In this paper, we shall consider the countable tightness of the subspaces of free (Abelian) groups over Lašnev spaces, which are the closed images of metric spaces.

A space X is of *countable tightness* if the closure of any subset A of X equals the union of closures of all countable subsets of A, and the tightness of a space X is denoted by $t(X)$. In [5], the authors showed that

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for a Lašnev space X, the tightness of $F(X)$ is countable if and only if X is separable or discrete. Moreover, in [7, Theorem 3.1], the authors showed that for a Lašnev space X, the tightness of $F(X)$ is countable if and only if the tightness of $F_8(X)$ is countable. Therefore, it is natural to pose the following question:

Question 1.1. *Let X be a Lašnev space. If the tightness of the subspace $F_n(X)$ is countable for some $n < 8$, is the tightness of $F(X)$ countable?*

In [13], K. Yamada proved that for an arbitrary metrizable space X, the following statements are equivalent: i), the subspace $A_4(X)$ is of countable tightness; ii), $A(X)$ is of countable tightness; and iii), the set of all non-isolated points of X is separable, and $A_3(X)$ is of countable tightness. Therefore, the tightness of $A_3(\bigoplus_{\alpha<\omega_1} C_\alpha)$ is countable, and the tightness of $A_4(\bigoplus_{\alpha<\omega_1} C_\alpha)$ is uncountable, where each C_α is a copy of $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ being the subspace of Euclidean space $\mathbb{R}$. It is natural to pose the following question:

Question 1.2. *Let X be a Lašnev space. If the subspace $A_n(X)$ is of countable tightness for some natural number $n > 3$, is $A(X)$ of countable tightness?*

In section 3, we give an answer to Question 1.1 in a class of non-metrizable Lašnev spaces under some set theory hypothesis. In section 4, we give an answer to Question 1.2 in some class of Lašnev spaces, and prove that if X is the closed image of a locally separable metrizable space, then the tightness of $A_4(X)$ is countable if and only if the tightness of $A(X)$ is countable.

2. Preliminaries

In this section, we introduce the necessary notations and terminologies. First of all, let $\mathbb{N}$ be the set of all positive integers and ω the first infinite ordinal. For a space X, we always denote the set of all the non-isolated points in X by $\mathrm{NI}(X)$. For undefined notations and terminologies, the reader may refer to [2], [3], [4] and [9].

Let X be a topological space and $A \subseteq X$ be a subset of X. The *closure* of A in X is denoted by $\overline{A}$. Moreover, A is called *bounded* if every continuous real-valued function f defined on A is bounded. The space X is called a *k-space* provided that a subset $C \subseteq X$ is closed in X if $C \cap K$ is closed in K for each compact subset K of X. A subset P of X is called a *sequential neighborhood* of $x \in X$, if each sequence converging to x is eventually in P. A subset U of X is called *sequentially open* if U is a sequential neighborhood of each of its points. A subset F of X is called *sequentially closed* if $X \setminus F$ is sequentially open. The space X is called a *sequential space* if each sequentially open subset of X is open.

Let $\mathscr{P}$ be a family of subsets of a space X. Then, $\mathscr{P}$ is called a *k-network* if for every compact subset K of X and an arbitrary open set U containing K in X there is a finite subfamily $\mathscr{P}' \subseteq \mathscr{P}$ such that $K \subseteq \bigcup \mathscr{P}' \subseteq U$. Recall that X is an *$\aleph$-space* (resp. *$\aleph_0$-space*) if X has a σ-locally finite (resp. countable) k-network. We list two well-known facts about the Lašnev spaces as follows.

Fact 1. A Lašnev space is metrizable if it contains no closed copy of S_ω.

Fact 2. A Lašnev space is an $\aleph$-space if it contains no closed copy of S_{ω_1}.

Let X be a non-empty space. Throughout this paper, $X^{-1} = \{x^{-1} : x \in X\}$ and $-X = \{-x : x \in X\}$, which are just two copies of X. For every $n \in \mathbb{N}$, $F_n(X)$ denotes the subspace of $F(X)$ that consists of all words of reduced length at most n with respect to the free basis X. The subspace $A_n(X)$ is defined similarly. We always use $G(X)$ to denote topological groups $F(X)$ or $A(X)$, and $G_n(X)$ to $F_n(X)$ or $A_n(X)$ for each $n \in \mathbb{N}$. Therefore, any statement about $G(X)$ applies to $F(X)$ and $A(X)$, and $G_n(X)$ applies to $F(X)$ and $A(X)$. Let e be the neutral element of $F(X)$ (i.e., the empty word) and 0 be that of $A(X)$. For