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Embedding cones over trees into their symmetric products

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A B S T R A C T

Given a metric continuum X and a positive integer n, let $F_n(X)$ be the hyperspace of nonempty sets of X with at most n points and let $\text{Cone}(X)$ be the topological cone of X. We say that a continuum X is cone-embeddable in $F_n(X)$ if there is an embedding h from $\text{Cone}(X)$ into $F_n(X)$ such that $h(x, 0) = \{x\}$ for each x in X. In this paper, we characterize trees X that are cone-embeddable in $F_n(X)$.

1. Introduction

A *continuum* is a nonempty, compact, connected metric space. If X is a continuum and n is a positive integer, then $C(X)$ denotes the hyperspace of subcontinua of X and $F_n(X)$ the hyperspace of nonempty subsets of X with at most n points, both endowed with the Hausdorff metric. Given a topological space X, the *topological cone of* X, denoted by $\text{Cone}(X)$, is the quotient space obtained from $X \times [0, 1]$ by shrinking $X \times \{1\}$ to a point, called the *vertex of* $\text{Cone}(X)$, and denoted by ν_X. An element of $\text{Cone}(X)$, different from ν_X, will be denoted as an ordered pair (x, t), where $x \in X$ and $t \in [0, 1)$. For convenience, we will consider that the vertex ν_X of $\text{Cone}(X)$, is equal to any ordered pair of the form $(x, 1)$, where $x \in X$.

The problem of determining conditions under which $\text{Cone}(X)$ is homeomorphic (or topologically equivalent) to $C(X)$ has been studied by many authors. Moreover, some authors have studied the problem of characterizing those continua X for which there exists a finite-dimensional continuum Z such that $C(X)$ is homeomorphic to $\text{Cone}(Z)$.

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Recently, A. Illanes and V. Martínez-de-la-Vega have studied the problem of determining those continua X for which there exists a continuum Z such that $F_n(X)$ is homeomorphic to Cone(Z) ([5]). In particular, they consider the problem when X is a finite graph or a fan (a dendroid with exactly one ramification point) and Z is finite dimensional.

In the rest of the paper, for a continuum X, the symbol $\mathcal{H}(X)$ denotes either $C(X)$ or $F_n(X)$ for some $n \in \mathbb{N}$.

Definition 1. Let X be a continuum. We say that X is **cone-embeddable in** $\mathcal{H}(X)$ if there exists an embedding $h : \text{Cone}(X) \to \mathcal{H}(X)$ such that $h(x, 0) = \{x\}$ for each $x \in X$. Such embedding h will be called a **good embedding**.

The third author have studied the case $\mathcal{H}(X) = C(X)$, obtaining general properties of cone-embeddable continua in $C(X)$ ([8]), results on compactifications ([9]) and on arc-smooth continua ([10]). In this paper we work the case $\mathcal{H}(X) = F_n(X)$, when X is a tree and n is a positive integer.

2. Basics definitions and facts

In this section, we present basic concepts and the notation used through the paper. A *finite graph* is a continuum which can be written as the union of finitely many arcs, called *edges*, any two of which are either disjoint or intersect only in one or both of their end points. The *vertices* of a finite graph X, are the end points of the edges of X, the set of all vertices of X will be denoted by $V(X)$. A *tree* is a finite graph containing no simple closed curves. Throughout the paper, G will denote a finite graph and T will denote a tree. Given a positive integer k, a *simple k-od* is a finite graph which is the union of k arcs emanating from a single point, v, and otherwise disjoint from one another. The point v is called the *core* of the simple k-od.

Given a finite graph G, $p \in G$ and a positive integer k, we say that p *is of order* k, denoted by $\text{Ord}_G(p) = k$, if p has a closed neighborhood which is homeomorphic to a simple k-od having p as the core. If $\text{Ord}_G(p) = 1$, then p has a neighborhood which is an arc having p as one of its end points and we will call it an *end point of* G. If $\text{Ord}_G(p) = 2$, then p has a neighborhood which is an arc, p is not an end point of it, and we will call it an *ordinary point of* G. A point $p \in G$ is a *ramification point of* G if $\text{Ord}_G(p) \geq 3$. In order to distinguish the vertices from the rest of the points of a finite graph which is not a simple closed curve, we assume that each vertex of G is either an end point of G or a ramification point of G. With this restriction, the two end points of an edge of G may coincide, in whose case such an edge is a simple closed curve. This kind of edges will be called *loops*. We will denote by $\text{Ram}(G)$ the set of all ramification points of G and by $E(G)$ the set of all end points of G.

Given two different vertices $p, q \in G$, we say that p and q are *adjacent* provided that there exists an edge J of G such that p and q are the end points of J. Given a vertex $p \in G$, let

$$N_G(p) = \{q \in V(G) : q \text{ is adjacent to } p\}.$$

Note that for a point $p \in T$, $\text{Ord}_T(p) = |N_T(p)|$, where $|A|$ denotes the cardinality of the set A.

For each pair of distinct vertices $p, q \in T$ we will denote by $pq = [p, q]$ the only arc in T joining p and q, $[p, q)$ will be the set $\{x \in pq : x \neq q\}$. We have in a similar way the natural description of (p, q).

Let G be a finite graph and $p \in \text{Ram}(G)$. We denote by $L_G(p)$ the set of end points adjacent to p in G. We define the *ramification order of p in G* as $ro_G(p) = |N_G(p) - L_G(p)| = Ord_G(p) - |L_G(p)|$, the number of ramification points adjacent to p in G. For a tree T such that $|\text{Ram}(T)| \geq 2$ we say that a ramification point p of T is *exterior* if p is an end point of the finite graph $E_p = T - \bigcup\{(p, e] : e \in L_T(p)\}$ obtained by removing all end points adjacent to p together with their corresponding edges; i.e., if $ro_T(p) = 1$. We will denote by $\text{Ext}(T)$ the set of exterior points. Observe that the existence of a non exterior ramification point is equivalent to the fact that $|\text{Ram}(T)| \geq 3$.