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Pseudocompleteness in the category of locales ☆

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We define pseudocompleteness in the category of locales in a conservative way; so that a space is pseudocomplete in the sense of Oxtoby [24] if and only if the locale it determines is pseudocomplete. We show that a pseudocomplete locale whose G_δ-sublocales are complemented (for instance if it is scattered) is a Baire locale in the sense of Isbell [20]. Our main theorem is that products of pseudocomplete locales are pseudocomplete. Whereas every discrete space is pseudocomplete, and Boolean locales generalize discrete spaces, we demonstrate that not every Boolean locale is pseudocomplete. In [27] Pichardo-Mendoza asks whether pseudocompleteness (in topological spaces) is an invariant of closed irreducible maps. We answer this in the affirmative.

0. Introduction

In his study of Baire spaces [24], Oxtoby introduced a class of spaces that he called pseudocomplete. He showed that they are Baire, and also have some interesting invariance properties, including that products of pseudocomplete spaces are pseudocomplete. These spaces have also been studied by some other authors. In particular, Pichardo-Mendoza [27] asks if pseudocompleteness is an invariant of closed irreducible maps. The present paper owes its genesis from a desire to answer Pichardo-Mendoza's question.

A number of completeness properties have been extended conservatively to locales, and shown to be stable under finite, countable, or arbitrary products; depending on the property. For instance, the product of complete uniform locales is complete [18], the product of Cauchy complete nearness locales is Cauchy complete [17], and products of countably many Čech complete nearness frames are Čech complete [12]. The term used in [12] is "constrained"; but, as described there, these are nearness frames that generalize Čech complete frames.

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We define pseudocomplete locales in the same way that Oxtoby [24] introduced pseudocomplete spaces. After casting the definition – which is in terms of sublocales – in purely frame-theoretic language (Proposition 2.3), we observe that it is "conservative"; which is to say a topological space X is pseudocomplete if and only if the locale $\mathsf{Lc}(X)$ it induces is pseudocomplete (Corollary 2.5).

With the Axiom of Choice, every locally compact locale is spatial; so if we assume the Axiom of Choice then a locally compact regular locale is pseudocomplete because, as Oxtoby showed in [24], a regular locally compact space is pseudocomplete. We show in Corollary 2.6 that, in fact, even without assuming that locally compact locales are spatial, every locally compact regular locale is pseudocomplete. As in spaces, every Cauchy complete (hence, every complete) metric locale is pseudocomplete (Corollary 2.9). Of course the parenthetical result is spatial because every complete uniform locale with a countable basis of uniformity is spatial, whence it is the topology of a complete metric space, and completely metrizable spaces are pseudocomplete [24].

Isbell [20] calls a locale Baire if each of its nontrivial open sublocales is of second category. A pseudocomplete locale in which every G_δ-sublocale is complemented is Baire (Proposition 2.12). A consequence of this is that every pseudocomplete scattered locale (a locale is scattered if its dissolution is Boolean) is Baire (Corollary 2.13).

One of our main results is that products of pseudocomplete locales are pseudocomplete (Theorem 4.3). This theorem enables us to see that the smallest dense sublocale of $\mathsf{Lc}(\mathbb{Q})$ is a non-spatial non-pseudocomplete closed sublocale of a pseudocomplete locale (Example 4.4). A spatial example of a non-pseudocomplete closed subspace of a pseudocomplete space was given by Oxtoby [24]. Seeing that the smallest dense sublocale of $\mathsf{Lc}(\mathbb{Q})$ is a Boolean locale which is not pseudocomplete, it is natural to ask for when Boolean locales are pseudocomplete. We have not been able to settle this, but it is not difficult to show that atomic Boolean locales are pseudocomplete.

By first proving a stronger localic result (Theorem 5.1), we answer in the affirmative the question of Pichardo-Mendoza mentioned above (Corollary 5.2). We then observe that similar arguments show that pseudocompleteness in spaces is an inverse invariant of open irreducible maps (Corollary 5.4).

To conclude the introduction, we remark that Todd [32] has introduced a formally weaker notion of pseudocompleteness in spaces. It is still not known whether it is equivalent to Oxtoby's pseudocompleteness. We do not study the localic version of Todd's pseudocompleteness here. It will appear in the doctoral thesis of the third-named author.

1. Preliminaries

1.1. Frames, very briefly

We shall use the terms "frame" and "locale" interchangeably. We refer the reader to [21] and [26] for a detailed study of frames. Here we recall very briefly what we need. A frame L is a complete lattice such that the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. The frame of open sets of a topological space X will be denoted by $\mathfrak{O}X$; and the locale determined by X will be written as $\mathsf{Lc}(X)$. We denote by h_* the right adjoint of a frame homomorphism h. The rather below and the completely below relations are denoted by $\prec$ and $\prec\!\!\prec$, respectively.

A set $C \subseteq L$ is a *cover* of L if $\bigvee C = 1$. We write $\operatorname{Cov} L$ for the set of all covers of L. For $C \in \operatorname{Cov} L$ and $a \in L$, the element Ca of L is defined by

$$Ca = \bigvee \{c \in C \mid c \wedge a \neq 0\}.$$