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Continuity of Jones' function is not preserved under monotone mappings

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ABSTRACT

We show two metric continua X and Z and a monotone surjective mapping $f : X \to Z$ such that the Jones' function T is continuous for X, but it is not continuous for Z. This answers a question by D. P. Bellamy.

1. Introduction

A *continuum* is a nonempty compact connected metric space. Given a continuum X, we consider the hyperspaces:

$$2^X = \{A \subset X : A \text{ is closed and nonempty}\},$$

$$C(X) = \{A \in 2^X : A \text{ is connected}\},$$

$$F_1(X) = \{\{x\} \in 2^X : x \in X\}.$$

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These hyperspaces are endowed with the Hausdorff metric H ([5, Definition 2.1]).

Given $A \subset X$, let

$$T(A) = \{p \in X : \text{ for each } M \in C(X) \text{ with } p \in \text{ int}_X(M),\ M \cap A \neq \emptyset\}.$$

If $p \in X$, we write $T(p)$ instead of $T(\{p\})$. If necessary, we write T_X instead of T.

In [6], F. B. Jones introduced the function T, this function has been used to study a number of properties on continua.

The function T is *continuous* for X provided that its restriction $T|_{2^X} : 2^X \to 2^X$ is continuous. The continuum X is T-*additive* if $T(A) \cup T(B) = T(A \cup B)$ for every $A, B \in 2^X$.

In [1], D. P. Bellamy, studied properties that can deduced from the continuity of the function T.

It is easy to show that if X is a locally connected continuum, then $T|_{2^X}$ is the identity, so T is continuous for X.

If X is an indecomposable continuum, then $T(A) = X$ for each $A \subset X$, so T is continuous for X.

In [2, Remark 2, p. 10], D. P. Bellamy remarked that if X is the circle of pseudo-arcs, then T is continuous for X.

W. Lewis proved ([8]) that for each curve (1-dimensional continuum) M, there exists the respective curve X_M of pseudo-arcs. That is, X_M is a curve that has a terminal continuous decomposition into pseudo-arcs such that the decomposition space is homeomorphic to M. In [10], it was observed that if X is a locally connected curve of pseudo-arcs, then T is continuous for X.

So, we have the following families of continua for which T is continuous:

(a) locally connected continua,
(b) indecomposable continua,
(c) the locally connected curves of pseudo-arcs, and
(d) the monotone images of locally connected curves of pseudo-arcs (see Theorem 2.3 below).

As we can see, there are only few continua for which it is known that T is continuous. Observing these examples, one can see that the following problems by D. P. Bellamy are very natural.

Problem 1.1 *([3, Problem 161]).* If T is continuous for X, is it true that X is T-additive?

Problem 1.2 *([3, Problem 162]).* If T is continuous for X, is it true that the collection $\{T(p) : p \in X\}$ is a continuous decomposition of X such that the quotient space is locally connected?

Problem 1.3 *([3, Problem 163]).* If T is continuous for X and there is a point p in X such that $T(p)$, has nonempty interior, is X indecomposable?

Problem 1.4 *([7, 155]).* If T_X is continuous for X and $f : X \to Z$ is a monotone surjection, is it true that T_Z is continuous for Z?

In this paper we show some relationships among Problems 1.1–1.4 and we give a negative answer to Problem 1.4, by giving a continuum X for which T_X is continuous such that X contains an arc L with the property that if $Z = X/L$ is the space obtained by shrinking L to a point, then T_Z is not continuous for Z.

2. Relationships

A *mapping* is a continuous function. Given a continuum X, $p \in X$, $A \subset X$ and $\varepsilon > 0$, let $B(p, \varepsilon)$ denote the open ε-ball around p and $N(A, \varepsilon)$ is the union of all the ε-balls around points of A. The continuum X