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Homogeneity degree of fans ☆

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The homogeneity degree of a topological space X is the number of orbits of the action of the homeomorphism group of X on X. We initiate a study of dendroids of small homogeneity degree, beginning with fans. We classify all smooth fans of homogeneity degree 3, and discuss non-smooth fans and prove some results on degree 4.

1. Introduction

For a topological space X we denote by $\mathcal{H}(X)$ the group of homeomorphisms of X onto itself. Given $x \in X$, we denote by $\mathrm{Orb}_X(x)$ the orbit of x under the action of $\mathcal{H}(X)$ on X; that is, $\mathrm{Orb}_X(x) = \{h(x) : h \in \mathcal{H}(X)\}$. The *homogeneity degree* of X is the number of orbits for the action of $\mathcal{H}(X)$ on X. Alternatively, we say X is $\frac{1}{n}$-*homogeneous* to mean that X has homogeneity degree n.

A space is *homogeneous* if its homogeneity degree is 1. Homogeneity is a classical and well-studied notion in continuum theory; however, some important classes of spaces (e.g. dendroids) include no homogeneous spaces, and thus do not interact with this theory. In this paper, we aim to contribute evidence that an appropriate notion of homogeneity for dendroids, especially for fans, is $\frac{1}{3}$-homogeneous.

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The study of $\frac{1}{n}$-homogeneous spaces was formally started in 1989 by H. Patkowska, who defined this term and gave a classification of $\frac{1}{2}$-homogeneous polyhedra (see [24]). Prior to this, without the use of such a term, J. Krasinkiewicz proved in 1969 [17] that the universal Sierpiński curve is $\frac{1}{2}$-homogeneous.

Recent work on $\frac{1}{n}$-homogeneous continua began in 2006 with classifications of the arc and circle in terms of $\frac{1}{2}$-homogeneity of the hyperspace of subcontinua, given in [21]. That paper initiated a series of works on homogeneity degree of hyperspaces, as well as cones and suspensions of continua, primarily focused on $\frac{1}{2}$-homogeneity.

Besides hyperspaces, other recent work on $\frac{1}{n}$-homogeneous continua has also been largely focused on the case $n = 2$. In [4], it is shown that the only $\frac{1}{2}$-homogeneous chainable continua with two endpoints are the arc, and the arc of pseudo-arcs with the two ends collapsed to points. Cutpoints of $\frac{1}{2}$-homogeneous continua are studied in [22] and [23], and in [23] the authors prove that the arc is the only $\frac{1}{2}$-homogeneous hereditarily decomposable continuum all of whose proper subcontinua are chainable. Examples of $\frac{1}{2}$-homogeneous indecomposable circle-like continua are given in [5] and [25], and $\frac{1}{n}$-homogeneous such examples, for $n \geq 3$, are constructed in [16].

An emerging theme in the study of continua, especially 1-dimensional continua, is that spaces of low homogeneity degree tend to be rare and remarkable – particularly those with some other interesting topological properties such as hereditary unicoherence, non-local connectivity, indecomposability, or planarity, for example. This theme can be seen in the above mentioned work and in other recent work, e.g. in the classification of all homogeneous plane continua completed in [15], and in the classification of all $\frac{1}{3}$-homogeneous dendrites [3].

Beyond dendrites, a natural step to take to explore more interesting continua of low homogeneity degree is to consider continua that are not locally connected, which leads us to consider dendroids. In this paper we initiate a study of dendroids of low homogeneity degree, beginning with fans. We classify all $\frac{1}{3}$-homogeneous smooth fans, and prove that there are no smooth fans with homogeneity degree 4. We also provide some necessary conditions for non-smooth fans to have homogeneity degrees 3 or 4. It is not yet known whether these exist – see Problem 1.

1.1. Definitions and notation

For a topological space X and $A \subset X$ the symbol $\mathrm{Cl}_X(A)$ denotes the closure of A in X. In this paper, all spaces considered will be metric spaces, and the metric will always be denoted by d. When we refer to the distance or convergence of closed subsets of X, it will be understood that we are considering the Hausdorff metric.

A *continuum* is a compact connected metric space. An *arc* is a space homeomorphic to the interval $[0, 1] = I$. We call any space homeomorphic to the standard middle-third Cantor set a *Cantor set*. A *dendroid* is an arcwise connected and hereditarily unicoherent continuum. Given a dendroid X and points $p, q \in X$, we denote by pq the unique arc in X having p and q as its endpoints, and put $(pq) = pq - \{p, q\}$, $[pq) = pq - \{q\}$ and $(pq] = pq - \{p\}$.

For a dendroid X we say that p has *order* r (in the classical sense), in symbols $\mathrm{Ord}_X(p) = r$, if p is a common endpoint of exactly r arcs in X which are disjoint from one another beyond p. A point $p \in X$ is an *endpoint* of X if $\mathrm{Ord}_X(p) = 1$, a *ramification point* of X if $\mathrm{Ord}_X(p) > 2$, and an *ordinary point* if $\mathrm{Ord}_X(p) = 2$. We denote by $E(X)$, $R(X)$, and $O(X)$ the set of endpoints, the set of ramification points, and the set of ordinary points of X respectively. It is clear that any homeomorphism of X onto itself preserves these three sets.

A *fan* is a dendroid X with exactly one ramification point called the *top* of X. Note that in a fan X the sets $E(X)$, $O(X)$, and $R(X)$ are all nonempty, hence any fan has homogeneity degree greater than or equal to 3. If X is a fan with top t, then $X = \bigcup_{e \in E(X)} te$ and $te \cap td = \{t\}$ for any two distinct $e, d \in E(X)$. It is clear that $O(X)$ is dense in X. From [19, Theorem 7.5, p. 311], we have that $E(X)$ is a G_δ subset of X.