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Algebrability of $\mathcal{S}$-continuous functions [☆]

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Considering the natural topology or $\mathcal{S}$-density topology on the domain and on the range we obtain different families of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$. We prove that certain differences of those families are strongly $\mathfrak{c}$-algebrable, i.e., they contain $\mathfrak{c}$-generated free subalgebras. In particular, we obtain strengthenings of recent results of J. Hejduk, A. Loranty, R. Wiertelak, S. Lindner and M. Terepeta.

1. Introduction

1.1. Algebrability and exponential-like method

In the last decades there appeared a new way of looking at largeness of subsets of certain algebraic structures. The idea is that, given an algebraic structure $\mathcal{A}$ (algebra, linear space, Banach space etc.) and its subset E, we can say that E is large, if it contains an (appropriately) large algebraic substructure. This idea can be formalized in many ways, and, indeed, there are many notions concerning this field. For a brief discussion we refer the reader to survey papers [2], [4], here we just mention that wide part of the studies deal with function spaces and their particular subsets. The reason is that function spaces are natural examples of

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algebraic structures, and functions with some strange properties have been investigated by mathematicians for decades. Our paper fits this branch of consideration.

We will deal with relatively strong notion – with the *strong κ-algebrability*.

Definition 1. Given a cardinal number κ and a subset E of an algebra $\mathcal{A}$, we say that E is *strongly κ-algebrable*, if $E \cup \{0\}$ contains a κ-generated free subalgebra (that is, the minimal system of generators of this subalgebra has cardinality κ, and can be chosen to be *free*).

Remark 1. *Recall that a subset A of a subalgebra $\mathcal{A}$ is free*, if every map $f : A \to \mathcal{A}$ can be extended to a homomorphism. Equivalently, if for every nonconstant polynomial $f(a_1, ..., a_n)$ and every distinct $x_1, ..., x_n \in A$, $f(x_1, ..., x_n) \neq 0$.

Recently, A. Bartoszewicz, M. Filipczak and M. Balcerzak in [1] developed a very useful method of proving strong $\mathfrak{c}$-algebrability in the algebra of all real functions $\mathbb{R}^{\mathbb{R}}$, which was successfully used in various problems (see also papers [3], [10] and references therein). We will also exploit this method.

We say that a function $f \colon \mathbb{R} \to \mathbb{R}$ is *exponential-like* if it is given by formula

$$f(x) = \sum_{i=1}^{m} \alpha_i e^{\beta_i x} \tag{1}$$

for some $\alpha_1, \ldots, \alpha_m \in \mathbb{R} \setminus \{0\}$ and distinct $\beta_1, \ldots, \beta_m \in \mathbb{R} \setminus \{0\}$.

Property 1 *([1]). Let $f \colon \mathbb{R} \to \mathbb{R}$ be an exponential-like function of the form (1). Then:*

a) *For every $c \in \mathbb{R}$, the preimage $f^{-1}(\{c\})$ has at most m elements.*
b) *There exist $x_0 < x_1 < \ldots < x_k$ such that f is strictly monotone on each interval (x_i, x_{i+1}), $i = 0, \ldots, k-1$ and $k \leq m$.*

Theorem 1 *([1]). (Exponential-like method) Given a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$, assume that there exists a function $F \in \mathcal{F}$ such that $f \circ F \in \mathcal{F} \setminus \{0\}$ for every exponential-like function $f \colon \mathbb{R} \to \mathbb{R}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable. More exactly, if $H \subset \mathbb{R}$ is a set of cardinality $\mathfrak{c}$, linearly independent over the rationals $\mathbb{Q}$, then e^{rF}, $r \in H$, are free generators of an algebra contained in $\mathcal{F} \cup \{0\}$.*

1.2. $\mathcal{S}$-density topology and continuous functions

By $\mathcal{L}$ we denote the family of Lebesgue measurable subsets of $\mathbb{R}$ and by λ we denote the Lebesgue measure. We write (x_n) for the sequence $(x_n)_{n \in \mathbb{N}}$. By $A \vartriangle B$ we denote the symmetric difference of sets A and B.

Recall that an operator $\Phi \colon \mathcal{L} \to \mathcal{L}$ is called a *lower density operator* if for every $A, B \in \mathcal{L}$, we have:

1) $\Phi(\emptyset) = \emptyset, \quad \Phi(\mathbb{R}) = \mathbb{R}$;
2) if $\lambda(A \vartriangle B) = 0$, then $\Phi(A) = \Phi(B)$;
3) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;
4) $\lambda(\Phi(A) \vartriangle A) = 0$.

If the last condition is replaced by

4′) $\lambda(\Phi(A) \setminus A) = 0$,