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A study on symmetric products of generalized metric spaces ☆

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A B S T R A C T

We study the relation between a space X satisfying certain generalized metric properties (for example, open (G), point-countable base, Collins–Roscoe property, semi-stratifiable, k-semistratifiable, semi-metrizable, scattered, point-countable cs-network, every compact set is metrizable) and its n-fold symmetric product $\mathcal{F}_n(X)$ satisfying the same properties. We also show that if X is an M_1-space then $\mathcal{F}(X)$ is an M_1-space, where $\mathcal{F}(X)$ is the hyperspace of finite subsets of X. A space X is a paracompact p-space if and only if its 2-fold symmetric product $\mathcal{F}_2(X)$ is a paracompact p-space. A Tychonoff space X is a Lindelöf Σ-space if and only if its 2-fold symmetric product $\mathcal{F}_2(X)$ is a Lindelöf Σ-space.

0. Introduction

There are many results on the hyperspace 2^X of nonempty closed subsets of a topological space X equipped with various topologies. Various subsets of 2^X are also widely studied. The following notations and notions follow from [21] and [12]. Given a space X, we define its *hyperspaces* as the following sets:

$2^X = \{A \subset X : A \text{ is closed and nonempty}\}$,

$\mathcal{C}(X) = \{A \in 2^X : A \text{ is compact}\}$,

$\mathcal{F}_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\}$, where n is a positive integer,

$\mathcal{F}(X) = \{A \in 2^X : A \text{ is finite}\}$.

2^X is topologized by the *Vietoris topology* defined as the topology generated by $\mathcal{B} = \{\langle U_1, \ldots, U_k\rangle : U_1, \ldots, U_k$ are open subsets of X, k is a positive integer$\}$, where $\langle U_1, \ldots, U_k\rangle = \{A \in 2^X : A \subset \bigcup_{j=1}^k U_j$ and

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$A \cap U_j \neq \emptyset$ for each $j \in \{1, \dots, k\}\}$. The topology on 2^X which is generated by $\mathcal{B}$ is also called the *finite topology* [21, Definition 1.7]. Note that, by definition, $\mathcal{C}(X), \mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are subsets of 2^X. Hence, they are topologized with the appropriate restriction of the Vietoris topology. $\mathcal{F}_n(X)$ is called the *n-fold symmetric product of* X [12] and $\mathcal{F}(X)$ is called *the hyperspace of finite subsets of* X [23, Abstract].

The following summary on results of $\mathcal{F}_n(X)$ is taken from [12, page 94]. The n-fold symmetric product $\mathcal{F}_n(X)$ of a space X, originally defined in 1931 by Borsuk and Ulam [5] is the quotient of X^n formed by the quotient map $(x_1, x_2, \dots, x_n) \mapsto \{x_1, x_2, \dots, x_n\}$. If X is a Hausdorff space and n is a positive integer, then $\mathcal{F}_n(X)$ is a closed subset of 2^X and the union of all symmetric products of X is the subspace $\mathcal{F}(X)$, which is dense in 2^X.

Mizokami presents a survey of results relating a generalized metric property of a space X with the hyperspace $\mathcal{C}(X)$ and $\mathcal{F}(X)$ [23]. In [12], Good and Macías studied symmetric products of generalized metric spaces. They considered several generalized metric properties and studied the relation between a space X satisfying such property and its n-fold symmetric product satisfying the same property.

In this note, we also study the relation between a space X satisfying certain generalized metric properties (for example, open (G), point-countable base, Collins–Roscoe property, regular G_δ-diagonal, semi-stratifiable, k-semistratifiable, semi-metrizable, scattered, point-countable cs-network, every compact set is metrizable) and its n-fold symmetric product satisfying the same properties. We also show that if X is an M_1-space then $\mathcal{F}(X)$ is an M_1-space. A space X is a paracompact p-space if and only if its 2-fold symmetric product $\mathcal{F}_2(X)$ is a paracompact p-space. A Tychonoff space X is a Lindelöf Σ-space if and only if its 2-fold symmetric product $\mathcal{F}_2(X)$ is a Lindelöf Σ-space.

All the spaces in this note are assumed to be Hausdorff. The set of all positive integers is denoted by $\mathbb{N}$ and ω is $\mathbb{N} \cup \{0\}$. Notations and terminology we follow [10] and [12].

1. On the n-fold symmetric product of a space

If $n \in \mathbb{N}$ and $\{U_i : 1 \leq i \leq n\}$ is a collection of subsets of a topological space X, then $\langle U_1, \dots, U_n \rangle$ denotes $\{A \in 2^X : A \subset \bigcup_{i=1}^n U_i$ and $A \cap U_i \neq \emptyset$ for each $i \in \{1, \dots, n\}\}$.

Remark 1. ([12, Remark 2.1]) Let X be a space and let n be an integer greater than or equal to two. Note that $\mathcal{F}_1(X)$ is closed in $\mathcal{F}_n(X)$ and $\xi : \mathcal{F}_1(X) \twoheadrightarrow X$ given by $\xi(\{x\}) = x$ is a homeomorphism.

The following notations are also taken from [12].

Notation 2. ([12, Notation 2.2]) Let X be a space and let n be a positive integer. To simplify notation, if $U_1, \dots, U_s$ are open subsets of X, then $\langle U_1, \dots, U_s \rangle_n$ denotes the intersection of the open set $\langle U_1, \dots, U_s \rangle$ of the Vietoris Topology, with $\mathcal{F}_n(X)$.

Notation 3. ([12, Notation 2.3]) Let X be a space and let n be a positive integer. If $\{x_1, \dots, x_r\}$ is a point of $\mathcal{F}_n(X)$ and $\{x_1, \dots, x_r\} \in \langle U_1, \dots, U_s \rangle_n$, then for each $j \in \{1, \dots, r\}$, we let $U_{x_j} = \bigcap\{U \in \{U_1, \dots, U_s\} : x_j \in U\}$.

Observe that $\langle U_{x_1}, \dots, U_{x_r} \rangle_n \subset \langle U_1, \dots, U_s \rangle_n$ [21, 2.3.1].

In [8, page 637], Collins and Roscoe introduce the following condition:

(G) for each $x \in X$, there is assigned a countable collection $\mathcal{G}(x)$ of subsets of X such that, whenever $x \in U$, U open, there is an open set $V(x, U)$ with $x \in V(x, U) \subset U$ such that whenever $y \in V(x, U)$ then $x \in N \subset U$ for some $N \in \mathcal{G}(y)$.

If a space X satisfies (G), then X is said to have the *Collins–Roscoe property* [26, Definition 2.1]. If X satisfies (G) and every element of $\mathcal{G}(x)$ is open in X for each $x \in X$, then X is said to satisfy *open (G)* [7, page 241].