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Topological difference of the iterated remainders

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ABSTRACT

Repeat taking remainders of Stone–Čech compactifications of the rationals $\mathbb{Q}$, the irrationals $\mathbb{P}$, and the Sorgenfrey line $\mathbb{S}$,

$$\mathbb{Q}^* = \beta\mathbb{Q}\backslash\mathbb{Q},\ \ \mathbb{Q}^{**} = \beta\mathbb{Q}^*\backslash\mathbb{Q}^*,\ \ \mathbb{Q}^{***}, \cdots ;$$

$$\mathbb{P}^*,\ \ \mathbb{P}^{**},\ \ \mathbb{P}^{***}, \cdots ;\ \ \mathbb{S}^*,\ \ \mathbb{S}^{**},\ \ \mathbb{S}^{***}, \cdots .$$

We show that all of these spaces are topologically different; the case of $\mathbb{Q}$ answers van Douwen's question [2].

1. Introduction

Consider the space of rationals $\mathbb{Q}$, and repeat taking its remainders of Stone–Čech compactifications $\mathbb{Q}^{(n+1)} = (\mathbb{Q}^{(n)})^* = \beta\,\mathbb{Q}^{(n)}\backslash\mathbb{Q}^{(n)}$ $(n \geqslant 0)$ where $\mathbb{Q}^{(0)} = \mathbb{Q}$, i.e.,

$$\mathbb{Q}^{(1)} = \mathbb{Q}^*,\ \ \mathbb{Q}^{(2)} = \mathbb{Q}^{**},\ \ \mathbb{Q}^{(3)} = \mathbb{Q}^{***}, \cdots .$$

Van Douwen [2] asked whether or not $\mathbb{Q}^{(n)} \approx \mathbb{Q}^{(n+2)}$ for $n \geqslant 1$, remarking that $\mathbb{Q}^{(m)}$ for even m is never homeomorphic to $\mathbb{Q}^{(n)}$ for odd n, because the former is σ-compact but the latter is (Čech-complete) not σ-compact. In this paper we will show that no perfect irreducible map $\mathbb{Q}^{(n)} \to \mathbb{Q}^{(n+2)}$ exists for any $n \geqslant 1$, answering van Douwen's question. We generally investigate the remainders of nowhere locally compact spaces, including the irrationals $\mathbb{P}$ and the Sorgenfrey line $\mathbb{S}$. The partial result $\mathbb{Q}^{(1)} \not\approx \mathbb{Q}^{(3)}$ was proved in [6], where we pointed out also that $\mathbb{Q}^{(n)}$ and $\mathbb{Q}^{(n+2)}$ have a similar structure of fiber-bundle.

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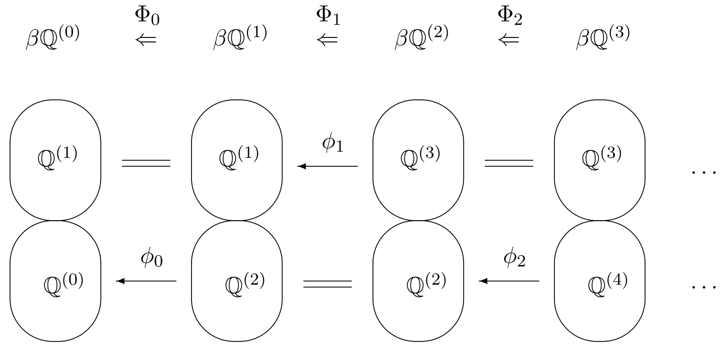

Fig. 1. The connections of the remainders $\mathbb{Q}^{(n+1)} = (\mathbb{Q}^{(n)})^* = \beta\,\mathbb{Q}^{(n)} \setminus \mathbb{Q}^{(n)}$ ($n \geqslant 0$) where $\mathbb{Q}^{(0)} = \mathbb{Q}$.

The precise connections of the remainders can be seen by the following construction. View $\beta\mathbb{Q}$ as a compactification of $\mathbb{Q}^{(1)}$, and let

$$\Phi_0 : \beta\mathbb{Q}^{(1)} = \mathbb{Q}^{(1)} \cup \mathbb{Q}^{(2)} \to \mathbb{Q} \cup \mathbb{Q}^{(1)} = \beta\mathbb{Q}$$

be the Stone extension of the identity map $id : \mathbb{Q}^{(1)} \to \mathbb{Q}^{(1)}$. Denote by

$$\phi_0 : \mathbb{Q}^{(2)} \to \mathbb{Q}^{(0)}$$

the restriction of Φ_0. Next let

$$\Phi_1 : \beta\mathbb{Q}^{(2)} = \mathbb{Q}^{(2)} \cup \mathbb{Q}^{(3)} \to \mathbb{Q}^{(1)} \cup \mathbb{Q}^{(2)} = \beta\mathbb{Q}^{(1)}$$

be the Stone extension of the identity map $id : \mathbb{Q}^{(2)} \to \mathbb{Q}^{(2)}$, and let

$$\phi_1 : \mathbb{Q}^{(3)} \to \mathbb{Q}^{(1)}$$

denote the restriction of Φ_1. In this way, for every $n \geqslant 0$ we can generally get the Stone extension

$$\Phi_n : \beta\mathbb{Q}^{(n+1)} = \mathbb{Q}^{(n+1)} \cup \mathbb{Q}^{(n+2)} \to \mathbb{Q}^{(n)} \cup \mathbb{Q}^{(n+1)} = \beta\mathbb{Q}^{(n)}$$

of the identity map $id : \mathbb{Q}^{(n+1)} \to \mathbb{Q}^{(n+1)}$, and its restriction

$$\phi_n : \mathbb{Q}^{(n+2)} \to \mathbb{Q}^{(n)}.$$

See Fig. 1. Since every Φ_n ($n \in \omega$) is perfect, so is every ϕ_n. Hence every $\mathbb{Q}^{(n)}$ ($n \in \omega$) is Lindelöf since both $\mathbb{Q}^{(0)} = \mathbb{Q}$, $\mathbb{Q}^{(1)}$ are Lindelöf. We can also see that $\mathbb{Q}^{(n)}$ is σ-compact for even n, but $\mathbb{Q}^{(n)}$ is not for odd n, because $\mathbb{Q}^{(0)}$ is σ-compact but $\mathbb{Q}^{(1)}$ is not. Note that $\mathbb{Q}^{(1)}$ is a perfect pre-image of the irrationals $\mathbb{P}$ (see §5).

A collection $\mathcal{B}$ of nonempty open sets of X is called a π-*base* for X if every nonempty open set in X contains some member of $\mathcal{B}$. The minimal cardinality of such a π-base is called the π-*weight* of X. Note that any dense subspace of X has the same π-weight as X, and any space of countable π-weight is separable. Consequently, any dense subset of a space of countable π-weight is also of countable π-weight, and hence separable. So, all of $\beta\mathbb{Q}^{(n)}$, $\mathbb{Q}^{(n)}$ ($n \in \omega$) are of countable π-weight, and hence separable.

Recall that an onto map $g : X \to Y$ is called *irreducible* if every non-empty open subset U of X contains some fiber $g^{-1}(y)$, and it is well known and easy to see that