Topology and its Applications 231 (2017) 306–315

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – In honor of Professor Yukihiro Kodama on his 85th birthday

Scattered spaces and selections

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ARTICLE INFO

Article history:
Received 24 May 2016
Received in revised form 24 August 2016
Accepted 7 September 2016
Available online 2 October 2017

MSC:
54B20
54C65
54F05
54F65
54G12

Keywords:
Vietoris topology
Continuous selection
Scattered space

ABSTRACT

If the Vietoris hyperspace $\mathscr{F}(X)$ of the nonempty closed subsets of a regular space X has a continuous zero-selection, then so does $\mathscr{F}(Z)$ for every nonempty $Z \subset X$. The present paper deals with the inverse problem showing that X is a scattered space provided $\mathscr{F}(Z)$ has a continuous selection for every nonempty countable $Z \subset X$. This is obtained by showing that a crowded regular space X contains a copy of the rational numbers provided its Vietoris hyperspace $\mathscr{F}(X)$ has a continuous selection. Some related problems and applications are discussed as well.

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1. Introduction

All spaces in this paper are assumed T_1 topological spaces. Let X be a space, and $\mathscr{F}(X)$ be the set of all nonempty closed subsets of X. A map $f : \mathscr{F}(X) \to X$ is a *selection* for $\mathscr{F}(X)$ if $f(S) \in S$ for every $S \in \mathscr{F}(X)$, and f is *continuous* if it is continuous with respect to the Vietoris topology τ_V on $\mathscr{F}(X)$. Recall that τ_V is generated by all collections of the form

$$\langle \mathscr{V} \rangle = \left\{ S \in \mathscr{F}(X) : S \subset \bigcup \mathscr{V} \text{ and } S \cap V \neq \varnothing, \text{ whenever } V \in \mathscr{V} \right\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of X.

A space X is *scattered* if every nonempty subset $A \subset X$ has an element $p \in A$ which is isolated relative to A, i.e. p is an isolated point of A. Since each $x \in \overline{A} \setminus A$ is a non-isolated point of $\overline{A}$, a space X is scattered if every nonempty closed subset of X has an isolated point. A selection $f : \mathscr{F}(X) \to X$ is called

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https://doi.org/10.1016/j.topol.2017.09.023
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a *zero-selection* if $f(S)$ is an isolated point of S for every $S \in \mathscr{F}(X)$. Evidently, every scattered space has a zero-selection, and every space which has a zero-selection is scattered. Continuous zero-selections imply some interesting properties. For instance, it was shown in [7] that a compact space X is an ordinal space if and only if $\mathscr{F}(X)$ has a continuous zero-selection. Here, X is an *ordinal space* if it is an ordinal equipped with the open interval topology. For some related results in the non-compact case, the interested reader is referred to [1,2,6,10,14].

In the present paper, we are interested in another aspect of zero-selections. Following Michael [18], let $\mathscr{A}(X)$ be the collection of all nonempty subsets of a set X. For a space X, one can endow $\mathscr{A}(X)$ with the Vietoris topology in precisely the same way as this was done for $\mathscr{F}(X)$ [18], see also the next section. However, the resulting hyperspace $(\mathscr{A}(X), \tau_V)$ is not so interesting. For instance, it is a folklore result that $(\mathscr{A}(X), \tau_V)$ is a T_1-space if and only if X is discrete. Despite of this, $(\mathscr{A}(X), \tau_V)$ plays an interesting role for zero-selections. Indeed, in the next section we will show that $\mathscr{F}(X)$ has a continuous zero-selection if and only if $\mathscr{A}(X)$ has a continuous selection, Theorem 2.1. The idea behind this result is simple. Namely, if $f : \mathscr{F}(X) \to X$ is a zero-selection and $S \in \mathscr{A}(X)$, then $f\left(\overline{S}\right) \in S$. The converse is also evident, and for a continuous selection $g : \mathscr{A}(X) \to X$, the value $g(S)$ must be an isolated point of $S \in \mathscr{A}(X)$, Corollary 2.5. These considerations reveal a natural relationship between zero-selections and the closure operator $\mathscr{cl}(S) = \overline{S} \in \mathscr{F}(X)$, for $S \in \mathscr{A}(X)$, see Theorem 2.1. An interesting aspect of this relationship is that to each zero-selection f for $\mathscr{F}(X)$ and $Z \in \mathscr{A}(X)$, one can associate the zero-selection $f \circ \mathscr{cl} \restriction \mathscr{F}(Z)$ for $\mathscr{F}(Z)$, Corollary 2.3. In this construction, the continuity of the selection is preserved as far as X is regular, see Corollary 2.6. Thus, for a regular space X with a continuous zero-selection for $\mathscr{F}(X)$, each hyperspace $\mathscr{F}(Z)$ for $Z \in \mathscr{A}(X)$, has a continuous selection, Theorem 2.1.

The substantial part of this paper deals with the inverse problem. Briefly, in Section 4 we show that a regular space X with a continuous selection for $\mathscr{F}(X)$ is scattered if and only if $\mathscr{F}(Z)$ has a continuous selection for each countable $Z \in \mathscr{A}(X)$, Theorem 4.1. This is based on an interesting result that each crowded regular space X with a continuous selection for $\mathscr{F}(X)$, contains a copy of the rational numbers, Theorem 3.1. In the last Section 5 of the paper, various disconnectedness-like properties are related to continuous selections for hyperspaces $\mathscr{F}(Z)$, for special subsets $Z \subset X$, see e.g. Corollary 5.3 and Theorem 5.4.

2. Continuous zero-selections

For a space X, as in [18], we will consider $\mathscr{A}(X)$ endowed with the Vietoris topology τ_V, i.e. with the topology generated by the basic τ_V-open sets of the form

$$\langle \mathscr{V} \rangle = \left\{ S \in \mathscr{A}(X) : S \subset \bigcup \mathscr{V} \text{ and } S \cap V \neq \varnothing,\ V \in \mathscr{V} \right\},$$

for $\mathscr{V}$ being a finite family of open subsets of X. In this section, we are interested in the *closure operator*

$$\mathscr{A}(X) \ni A \xrightarrow{\mathscr{cl}} \mathscr{cl}(A) = \overline{A} \in \mathscr{F}(X)$$

as a map from the hyperspace $(\mathscr{A}(X), \tau_V)$ to the hyperspace $(\mathscr{F}(X), \tau_V)$. This operator is naturally related to the existence of continuous zero-selections as the following theorem asserts.

Theorem 2.1. *Let X be a regular space. Then*

(a) *$\mathscr{F}(X)$ has a continuous zero-selection if and only if $\mathscr{A}(X)$ has a continuous selection.*
(b) *If $f : \mathscr{F}(X) \to X$ is a continuous zero-selection, then $f \circ \mathscr{cl} \restriction \mathscr{F}(Z)$ is a continuous zero-selection for $\mathscr{F}(Z)$, for every $Z \in \mathscr{A}(X)$.*

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