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Extension of functions and metrics with variable domains ☆

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ABSTRACT

Let (X, d) be a complete, bounded, metric space. For a nonempty, closed subset A of X denote by $C^*(A \times A)$ the set of all continuous, bounded, real-valued functions on $A \times A$. Denote by

$$C^\dagger = \bigcup \{C^*(A \times A) \mid A \text{ is a nonempty closed subset of } X\}$$

the set of all partial, continuous and bounded functions. We prove that there exists a linear, regular extension operator from $C^\dagger$ endowed with the topology of convergence in the Hausdorff distance of graphs of partial functions to the space $C^*(X \times X)$ with the topology of uniform convergence on compact sets. The constructed extension operator preserves constant functions, pseudometrics, metrics and admissible metrics. For a fixed, nonempty, closed subset A of X the restricted extension operator from $C^*(A \times A)$ to $C^*(X \times X)$ is continuous with respect to the topologies of pointwise convergence, uniform convergence on compact sets and uniform convergence considered on both $C^*(A \times A)$ and $C^*(X \times X)$.

1. Introduction

The problem of extending continuous functions has a long history and is fundamental in topology and analysis. Improvements to the Tietze extension theorem and its counterpart for metrics, the Hausdorff extension theorem [9], have been made by many authors. The Dugundji result [7] on continuous, linear

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extensions of partial, continuous functions defined on a closed subspace of a metric space has counterparts for metrics as well. Initially the problem of existence of continuous, linear operators extending (pseudo)metrics was raised and solved for some special cases by Bessaga [6] and was completely solved by the first named author [2] (see also [3] and [18]). In [3] the authors obtained a rather general result on linear, continuous extensions of all functions and metrics defined on a stratifiable space with a fixed domain.

Further generalizations of known results on extensions of functions are related to the problem of simultaneous extension of partial functions with variable domains. Stepanova [16] obtained a non-linear, continuous extension operator for continuous, real-valued functions defined on variable, compact subsets of a metric space. She proved that the restriction onto metric spaces is essential. Künzi and Shapiro [11] improved Stepanova's result by showing existence of continuous, linear and regular extension operators under the same assumptions. A generalization of the Künzi–Shapiro result for the noncompact domain case was obtained in [10] (see also [4]). The third and the fourth named authors constructed an analogue of the Künzi–Shapiro theorem for (pseudo)metrics [17]. They described a linear, regular operator extending continuous (pseudo)metrics defined on closed subsets of a compact metrizable space. This operator is continuous with respect to the Hausdorff metric topology on the set of partial (pseudo)metrics where every (pseudo)metric is identified with its graph. Note that the Hausdorff metric convergence of graphs of continuous functions with common domain implies pointwise convergence as well as uniform convergence on compact sets but does not imply the uniform convergence of these functions. However, if the limit function is uniformly continuous then this last implication is true (see [5] and [13]). Of course all the metrics considered in [17] are uniformly continuous because the initial space is compact, so the convergence of graphs of metrics defined on the whole space is equivalent to the uniform convergence.

In [15] the second and the third named authors obtained a counterpart of the main result from [17] for noncompact, complete, metric spaces X. They proved existence of a linear, regular operator extending bounded, continuous pseudometrics defined on closed, bounded subsets of X. The set of partial pseudometrics was endowed with the Hausdorff metric topology and the set of pseudometrics defined on X was considered with the topology of uniform convergence on compact sets. It was shown that the operator is continuous. However, the authors were not able to modify this operator to preserve metrics without losing regularity.

In the current paper we construct an improvement of the operator from [15]. Using ideas from [3], [17] and [15] we describe an extension operator for partial, continuous, bounded functions and (pseudo)metrics on variable, closed subsets of a complete, metric space with a broad set of properties. In particular, we show that our operator is linear, regular and continuous in several important topologies. Another important feature of our operator is that it preserves admissible metrics, a fact that is not generally true for the operator in [15].

2. Preliminaries

Let (X, d) be a complete, bounded, metric space with $\mathrm{card}(X) \geq 2$. We assume that $d(x, y) \leq 1$ for all $x, y \in X$. Denote by $\exp(X)$ the set of all nonempty, closed subsets of X endowed with the Hausdorff metric d_H generated by d. Recall that the Hausdorff distance between any sets $A, B \in \exp(X)$ is given by

$$d_H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}.$$

Denote the set of singleton subsets of X by $\exp_1(X)$.

For each $A \in \exp(X)$ let $C^*(A \times A)$ be the set of all continuous and bounded real-valued functions on $A \times A$. Define

$$C^\dagger = \bigcup\{C^*(A \times A) \mid A \in \exp(X)\}$$