Topology and its Applications 231 (2017) 386–399

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – In honor of Professor Yukihiro Kodama on his 85th birthday

Insertion of poset-valued maps with the way-below and -above relations

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ARTICLE INFO

Article history:
Received 30 July 2016
Received in revised form 31 March 2017
Accepted 3 April 2017
Available online 29 September 2017

MSC:
06B35
54C03
54D15
54D20

Keywords:
Insertion
Semi-continuous
Poset
Bicontinuous
Lattice
Way below
Way above
Countably paracompact
Normal
Paracompact
Dowker–Katětov insertion theorem
Bi-Scott topology
Bi-bounded complete

ABSTRACT

In this paper, with the way-below relation $\ll$ and the way-above relation $\ll_d$, we give a poset-valued insertion theorem by a pair of semi-continuous maps. Moreover, we show a poset-valued insertion theorem by a continuous map as follows: Let X be a paracompact Hausdorff space, P a bi-bounded complete, bicontinuous, pathwise connected, topological poset. For each upper semi-continuous map $f : X \to P$ with a lower bound and each lower semi-continuous map $g : X \to P$, if $\langle f, g\rangle$ has interpolated points pointwise, there exists a continuous map $h : X \to P$ such that $f \ll_d h \ll g$. A generalized Dowker–Katětov's insertion theorem, by using the way-below and -above relations on bicontinuous posets P, is also given.

1. Introduction

Throughout this paper, let $\mathbb{R}$ be the set of all real numbers, $\mathbb{N}$ the set of all natural numbers, and κ an infinite cardinal. All topological spaces are assumed to be T_1-spaces.

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[1] Supported by KAKENHI (No. 23540100).

https://doi.org/10.1016/j.topol.2017.09.035

Katětov–Tong's insertion theorem ([10], [15]) shows that a topological space X is normal if and only if for each upper semi-continuous function $f : X \to \mathbb{R}$ and each lower semi-continuous function $g : X \to \mathbb{R}$ with $f \leq g$ there exists a continuous function $h : X \to \mathbb{R}$ such that $f \leq h \leq g$. Another classical insertion theorem, Dowker–Katětov's insertion theorem ([2], [10]), says that a topological space X is normal and countably paracompact if and only if for each upper semi-continuous function $f : X \to \mathbb{R}$ and each lower semi-continuous function $g : X \to \mathbb{R}$ with $f < g$ there exists a continuous function $h : X \to \mathbb{R}$ such that $f < h < g$. Here, for maps $f, g : X \to \mathbb{R}$, the symbol $f \leq g$ (resp. $f < g$) stands for $f(x) \leq g(x)$ (resp. $f(x) < g(x)$) for every $x \in X$. Various extensions of Katětov–Tong's and Dowker–Katětov's insertion theorems are known. For maps to Banach lattices and ordered topological vector spaces, see [1], [9], [14], [16], [17] and [18]. On the other hand, various important insertion theorems had been given in case that the range Y of maps are not necessarily vector spaces. See [19] for complete chains Y, [4] for LOTS Y, [8] and [12] for continuous lattices or quasi-uniform spaces Y, [13] for completely distributive lattices Y, [6] for lattices or a hedgehog Y, and [7] for bounded complete domains Y, etc.

In this paper, we provide new insertion theorems for maps with values in bi-bounded complete and bicontinuous posets, which are not necessarily vector spaces or lattices, by using the way-below relation $\ll$ and the way-above relation $\ll_d$. The way-below and -above relations $\ll$ and $\ll_d$ are quite fundamental and useful for the study of order theory, topology, category theory and computer sciences, etc (see [5], [11] etc.).

In Section 2, we give preliminary facts and definitions. A lattice L endowed with a topology is said to be a *topological lattice* if the meet operation $\langle x, y\rangle \mapsto x \wedge y$ and the join operation $\langle x, y\rangle \mapsto x \vee y$; $L \times L \to L$, are continuous ([5]). Arranging this notion to a poset endowed with a topology, we introduce a new notion called a topological poset and give its basic property in Section 2.

Various definitions of lower and upper semi-continuity of maps $f : X \to P$ from a topological space X into a poset P are known. In Section 3, among them, we adopt the familiar definition by using $f = f_*$ and $f = f^*$, where a similar definition by using $f = f_*$ for maps with valued in bounded complete domains was formulated by Gutiérrez García, Kubiak and de Prada Vicente [7]. We also describe lower or upper semi-continuity with the way-below and way-above relations $\ll$ and $\ll_d$.

In Section 4, we give insertion theorems by a pair of upper and lower semi-continuous maps. We call a poset P *lower-bounded complete* (resp. *upper-bounded complete*) if every non-empty subset A of P with a lower bound (resp. an upper bound) has the inf (resp. sup). When P is lower-bounded complete and upper-bounded complete, we call P *bi-bounded complete*. Note that every bounded complete domain in the sense of [5] is bi-bounded complete. For maps $f, g : X \to P$ into a poset P, the symbol $f \ll g$ (resp. $f \ll_d g$, $f \leq g$) stands for $f(x) \ll g(x)$ (resp. $f(x) \ll_d g(x)$, $f(x) \leq g(x)$) for each $x \in X$. For a point z and a pair of points $\langle y, y'\rangle$ of a poset P, z is an *interpolated point of* $\langle y, y'\rangle$ if $y \ll_d z \ll y'$. A pair $\langle f, g\rangle$ of maps $f, g : X \to P$ *has interpolated points pointwise* if $\langle f(x), g(x)\rangle$ has an interpolated point for each $x \in X$.

Theorem 1.1. *Let X be a paracompact Hausdorff space, and P a bi-bounded complete, bicontinuous poset. For each upper semi-continuous map $f : X \to P$ and each lower semi-continuous map $g : X \to P$, if $\langle f, g\rangle$ has interpolated points pointwise, there exist an upper semi-continuous map $u : X \to P$ and a lower semi-continuous map $\ell : X \to P$ such that $f \ll_d \ell \leq u \ll g$.*

When a poset P is a topological space, we give another insertion theorem by using continuous maps as follows.

Theorem 1.2. *Let X be a paracompact Hausdorff space, P a bi-bounded complete, bicontinuous, pathwise connected, topological poset. For each upper semi-continuous map $f : X \to P$ with a lower bound $\perp_f$ and each lower semi-continuous map $g : X \to P$, if $\langle f, g\rangle$ has interpolated points pointwise, there exists a continuous map $h : X \to P$ such that $f \ll_d h \ll g$.*