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On a family of triangle groups in complex hyperbolic geometry

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In this paper, we study the deformation of triangle groups of type $(3, n, \infty)$ determined by three $\mathbb{R}$-circles R_0, R_1, R_2 with only rotational symmetry, which generalizes the problem studied by Falbel and Parker in [4].

1. Introduction

The deformation problem has been widely studied in geometry and representation theory. Given a discrete subgroup Γ of a Lie group G and a larger Lie group $H \supset G$, whether or not Γ admits any deformations into G, i.e. any representations of Γ into G which are close to the inclusion $\iota : \Gamma \hookrightarrow G \hookrightarrow H$ but not conjugate to it, and if so whether or not these remain discrete and faithful in some neighborhood of ι.

In complex hyperbolic geometry, the deformation of certain triangle groups of isometries of the hyperbolic plane $\mathbf{H}^2_{\mathbb{R}}$ into the isometry group of the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ has profound and fascinating results. There are two ways to represent complex triangle groups, the generators can be anti-holomorphic isometries or complex reflections. The (p, q, r)-group is the abstract group presented by

$$\langle \iota_0, \iota_1, \iota_2 : \iota_0^2 = \iota_1^2 = \iota_2^2 = (\iota_1\iota_2)^p = (\iota_0\iota_2)^q = (\iota_0\iota_1)^r = 1 \rangle.$$

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In [5], Goldman and Parker studied the deformation problem of ideal triangle groups which are parametrized by only one real parameter, Cartan's angular invariant, they partially classified which complex hyperbolic ideal triangle groups are discrete and faithful, and gave the famous Goldman–Parker conjecture.

In [10], Schwartz proved the Goldman–Parker conjecture: a representation ρ_s of the ideal triangle group is discrete and faithful if and only if $\rho_s(\iota_0\iota_1\iota_2)$ is not elliptic. Schwartz's ICM survey [12] gave an overview and a series of important conjectures about complex hyperbolic triangle group and suggested the trend future progress may take. He conjectured that representations should be discrete and faithful provided that both the images of $\iota_0\iota_1\iota_2$ and $\iota_0\iota_1\iota_2\iota_1$ are not elliptic. Obviously the Goldman–Parker conjecture is a special case of this conjecture. In [9], the authors studied the $(3, 3, n)$-group and also solved a special case of Schwartz's conjecture. We also refer readers to [8], [13].

If a discrete, faithful representation ρ is conjugate to a representation of abstract group Γ into $S(\mathbf{U}(1) \times \mathbf{U}(1,1)) < \mathbf{SU}(2,1)$, then it preserves a complex line and is called $\mathbb{C}$-Fuchsian. In [1], Falbel and Koseleff discussed the dimension of the deformation space of ideal triangle groups in complex hyperbolic space near a $\mathbb{C}$-Fuchsian representation, they showed the space contain a real four-dimensional ball. In [2], Falbel and Koseleff studied the same question for (p, q, ∞)-triangle groups in complex hyperbolic space, their main result showed that, if $2 < p \leq q$, then the deformation space contain open sets of dimension 0, 1 and 2 real dimension. Each open set of deformation space contains a $\mathbb{C}$-Fuchsian embedding. Moreover, if $p = 2$ then the deformation space contains open sets of dimensions 0 or 1. These results show some topological properties of the deformation space, but do not give the exactly description of the space.

In [4], Falbel and Parker constructed the space of discrete, faithful, type-preserving representations of the modular group $\Gamma = \mathbf{PSL}(2,\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$ into $\mathbf{Isom}(\mathbf{H}^2_{\mathbb{C}})$. The space contains six connected components: four isolated points and two intervals. This is the first Fuchsian group whose entire complex hyperbolic deformation space has been constructed. The components where both generators are complex reflections were studied as special cases in [2], there are four isolated points on the space. The component where the order two generator is reflection in a point and the order three generator is not a reflection was studied independently in [3] and [7]. The remaining component where the order two generator is reflection in a line and the order three generator is not a reflection was studied in [4]. They consider Γ be an index 2 subgroup of group $\widehat{\Gamma}$ generated by three involutions which correspond to reflections in the sides of a hyperbolic triangle with angles $\pi/2$, $\pi/3$ and 0. That is, $\widehat{\Gamma}$ is a $(2, 3, \infty)$-triangle group. Considering the subgroup $\hat{G} = \langle r_0 = \iota_1, r_1 = \iota_0\iota_2\iota_1\iota_2\iota_0, r_2 = \iota_2\iota_0\iota_1\iota_0\iota_2 = \iota_2\iota_1\iota_2\rangle \subset \hat{\Gamma}$ of index 6, they proved that

Theorem 1.1. *(Theorem 1.2 of [4]) Let R_0 be a totally real plane in $\mathbf{H}^2_{\mathbb{C}}$ and $g \in \mathbf{PU}(2,1)$ have order 3. Suppose that R_0, $R_1 = g(R_0)$ and $R_2 = g^2(R_0)$ are pairwise asymptotic. Let r_0, $r_1 = gr_0g^2$ and $r_2 = g^2r_0g$ be (antiholomorphic) involutions fixing R_0, R_1 and R_2. Also, suppose that there is a totally real plane R so that r, reflection in R, maps R_0 to itself and interchanges R_1 and R_2. Then the group generated by r_0, r_1 and r_2 is a discrete, faithful representation of $\hat{G}$ if and only if $(r_0r_1r_2)^2$ is not elliptic. In addition, it is type-preserving provided $(r_0r_1r_2)^2$ is loxodromic.*

They also conjectured that a similar result should be true when the configuration R_0, R_1 and R_2 only has rotational symmetry by g. Motivated by their work and aiming to explore more properties about complex hyperbolic triangle groups, in this paper we consider this problem and study the complex hyperbolic deformation of the $(3, n, \infty)$-triangle group, that is, the abstract group presented by $\Gamma' = \langle \iota_0, \iota_1, \iota_2 : \iota_0^2 = \iota_1^2 = \iota_2^2 = (\iota_0\iota_1)^n = (\iota_0\iota_2)^3 = 1 \rangle$.

We consider the case where both generators of order 3 and order n are not complex reflections and finite index subgroup $G' = \langle r_0 = \iota_1, r_1 = \iota_0\iota_2\iota_1\iota_2\iota_0, r_2 = \iota_2\iota_0\iota_1\iota_0\iota_2\rangle \subset \Gamma'$. As in [4], the representation constructed in our main result is parametrized by the Cartan angular invariant $\mathbb{A}$ (see the definition of $\mathbb{A}$ in next section). Our strategy is to use $\mathbb{R}$-spheres to construct explicit fundamental domain, then use Poincaré polyhedron theorem or combination theorem to get the discreteness. There does not exist an anti-holomorphic symmetry