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The decomposition uniqueness for infinite Cartesian products

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ABSTRACT

It is well known that the finite product of locally connected curves has the decomposition uniqueness property. It is natural to ask whether the same holds for infinite products. In general, this isn't the case – the Hilbert cube is homeomorphic to the countable infinite product of triods. We prove that if X is a product of locally connected curves then X has the decomposition uniqueness property if only finitely many of the factors are locally dendrites. The last condition is not necessary. It has been shown by Eberhart that the infinite torus has the decomposition uniqueness property.

1. Introduction

A space X is called *prime* if it is not homeomorphic to the Cartesian product of two spaces, each of them containing at least two points. Decomposing a space into Cartesian prime factors is, in general, not unique. For example, the products $[0,1] \times [0,1)$ and $[0,1) \times [0,1)$ are homeomorphic, but their respective factors aren't. The decomposition uniqueness does not hold in the compact case either. Borsuk [3] has constructed a countable family of different continua $\{X_i\}_{i\in\mathbb{N}}$ such that $X_i \times I$ are homeomorphic for every $i \in \mathbb{N}$.

There are a few affirmative results on the decomposition uniqueness into 1-dimensional factors. Recently, the author [14] has proved that the decomposition into a finite Cartesian product of locally connected curves is unique. On the other hand, Cauty [4] proved that every homeomorphism of a finite product of locally connected curves with sufficiently many circles is a homeomorphism product up to a permutation of the factors.

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It is well known that the Hilbert cube is homeomorphic to the countable infinite product of triods (see [1]). Thus, it is natural to ask whether the infinite product of more complicated curves, for instance, Menger-like curves, satisfies the decomposition uniqueness property.

The main theorem of this note reads as follows:

Theorem 1. *Let X be a product of locally connected curves. If only a finite number of the factors are local dendrites, then X has the decomposition uniqueness property.*

An important step toward proving our theorem is Theorem 2 due to Kennedy [12], which generalizes the result of Cauty [4] (see also [13] and [11]). Its statement is situated in Section 3.

The another ingredient is the concept of stable and labile points used previously in [10] and [15] and the idea of elementary sets in Cartesian products. This part of our work is contained in Section 4; here, the main result is Theorem 3, which was independently proved by the author and Furdzik [10]. The Theorem 3 is a natural generalization of the earlier results from [9] and [14]. The proof of Theorem 3 presented in Section 4 makes use of some ideas from [9] and [10], but in a simpler setting. The some tools were also used in [15]. More general version of Theorem 3 can be found in [5].

Finally, Section 5 contains the proof of Theorem 1.

2. Notation and tools

Our terminology follows that of [7] and [8].

If $X = \prod_{i\in A} X_i$ then, for every $i \in A$, the map $p_{X_i}\colon X \to X_i$ is the natural projection onto X_i and, for every $x \in X$, let $x_n = p_{X_n}(x)$. For every $(x_j)_{j\in A\setminus\{i\}} \in \prod_{j\in A\setminus\{i\}} X_j$, the map

$$r^i_{(x_j)_{j\in A\setminus\{i\}}} : X_i \to X$$

is defined by the formula

$$r^i_{(x_j)_{j\in A\setminus\{i\}}}(z) := (y_j)_{j\in A},$$

where

$$y_j := \begin{cases} x_j & \text{for} \quad j \in A \setminus \{i\} \\ z & \text{for} \quad j = i \end{cases}$$

A continuous mapping $h\colon X \times I \to X$ is a *homotopic deformation* if $h(x, 0) = x$ for every $x \in X$.

Definition 1. A point $x \in X$ is *stable* if, for every homotopic deformation h of the space X, we have $h(x, 1) = x$. A point $x \in X$ is *labile*, if for every $y \in X$, there exists a homotopic deformation h_y of the space X such that $h_y(x, 1) = y$.

We will denote the set of stable points by $S(X)$ and the set of labile points by $R(X)$.

Definition 2. Let $z \in X$. By z^X we denote the set of points $x \in X$ such that, there exists a homotopic deformation h of the space X satisfying $h(z, 1) = x$.

Remark 1. Using the definitions of the sets $S(X)$ and $R(X)$, it is easy to prove:

(1) The set $S(X)$ consists of the points $x \in X$ such that $x^X = \{x\}$,
(2) The set $R(X)$ consists of the points $x \in X$ such that $x^X = X$.