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More on products of Baire spaces

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ABSTRACT

New results on the Baire product problem are presented. It is shown that an arbitrary product of almost locally ccc Baire spaces is Baire; moreover, the product of a Baire space and a 1st countable space which is β-unfavorable in the strong Choquet game is Baire.

1. Introduction

A topological space is a *Baire space* provided countable intersections of dense open subsets are dense [13]. If the product $X \times Y$ is Baire, then X, Y must be Baire; however, the converse is not true in general. Indeed, Oxtoby [20] constructed, under CH, a Baire space with a non-Baire square, and various absolute examples followed (see [4], [8], [22], [23]). As a result, there has been a considerable effort to find various completeness properties for the coordinate spaces to get Baireness of the product (cf. [17], [10], [20], [13], [1], [29], [8], [22], [9], [31], [3], [19], [18]). There have been two successful approaches in solving the product problem: given Baire spaces X, Y, either one adds some condition to Y (such as 2nd countability [20], the uK-U property [9], having a countable-in-itself π-base [31]), or strengthens completeness of Y (to Čech-completeness, (strong) α-favorability [1], [29], or more recently, to hereditary Baireness [3], [19], [18]). It is the purpose of this paper to generalize these product theorems, as well as, show how a new fairly weak completeness property

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of *β-unfavorability in the strong Choquet game* [24], [6], [27] can be added to the list of spaces giving a Baire product.

Since Baire spaces can be characterized via the Banach–Mazur game, it is not surprising that topological games have been applied to attack the Baire product problem. Our results continue in this line of research (precise definitions will be given in the next section); in the games two players take countably many turns in choosing objects from a topological space X: in the *strong Choquet game* [2,14] player β starts, and always chooses an open set V and a point $x \in V$, then player α responds by choosing an open set U such that $x \in U \subseteq V$, next β chooses an open set V' and a point x' with $x' \in V' \subseteq U$, etc. Player α wins if the intersection of the chosen open sets is nonempty, otherwise, β wins.

The strong Choquet game provides a useful unifying platform for studying completeness-type properties, as the following two celebrated theorems demonstrate in a metrizable space X:

- $Ch(X)$ is α-favorable if and only if X is completely metrizable [2],
- $Ch(X)$ is β-unfavorable if and only if X is hereditarily Baire (i.e. the nonempty *closed* subspaces of X are Baire) [6,27,24].

The *Banach–Mazur game* $BM(X)$ [13] (also called the *Choquet game* [14]) is played as $Ch(X)$, except that both β, α choose open sets only. In a topological space X, $BM(X)$ is β-unfavorable iff X is a *Baire space* [21,16,26]; consequently, if $BM(X)$ is α-favorable, then X is a Baire space.

To put our results in perspective, recall that $X \times Y$ is a Baire space if X is a Baire topological space and

- either Y is a topological space such that $BM(Y)$ is α-favorable (in particular, if $Ch(Y)$ is α-favorable) [29],
- or Y is a hereditarily Baire space which is metrizable [19], or more generally, 1st countable T_3 space [18].

Since there are spaces which are α-favorable in the strong Choquet game but are not hereditarily Baire (the Michael line [7]), as well as metric hereditarily Baire spaces, which are not α-favorable in the Banach–Mazur game (a Bernstein set [7]), being β-unfavorable in the strong Choquet game is distinct from both hereditary Baireness as well as being α-favorable in the strong Choquet game, thus, it is natural to ask the status of this property in the Baire product problem. Our main result in Section 3 (Theorem 3.2) implies the following:

Theorem 1.1. *Let X be a Baire space, Y be a 1st countable topological space such that $Ch(Y)$ is β-unfavorable. Then $X \times Y$ is a Baire space.*

The proof works for finite products, but it does not naturally extend to infinite products, so we will separately consider the infinite product case in Section 4, using the idea of a *Krom space* ([15], [11]), to obtain:

Theorem 1.2. *Let I be an index set, and X_i be an almost locally ccc Baire space (defined in Section 3) for each $i \in I$. Then $\prod_i X_i$ is a Baire space.*

2. Preliminaries

Unless otherwise noted, all spaces are topological spaces. As usual, ω denotes the non-negative integers, and $n \geq 1$ will be considered as sets of predecessors $n = \{0, \ldots, n-1\}$. Let $\mathcal{B}$ be a base for a topological space X, and denote

$$\mathcal{E} = \mathcal{E}(X) = \mathcal{E}(X, \mathcal{B}) = \{(x, U) \in X \times \mathcal{B} : x \in U\}.$$