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Weak well-filtered spaces and coherence

Chongxia Lu, Qingguo Li *

College of Mathematics and Econometrics, Hunan University, Changsha, Hunan, 410082, PR China

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It was shown in [6] that a well-filtered dcpo L is coherent in its Scott topology if and only if for every $x, y \in L$, $\uparrow x \cap \uparrow y$ is compact in the Scott topology. We generalize this result to weak well-filtered posets which are defined in this paper. We obtain that every weak well-filtered poset is always a consistent dcpo. We separate the class of weak well-filtered dcpos from the well-filtered ones by showing that Johnstone's example of a non-sober dcpo belongs to the former but not the latter class.

1. Introduction

Coherence is an important property in domain theory due to its role in the categorical classification of continuous domains [1,3]. Jung obtained in [7] that coherence is equivalent to Lawson compactness on pointed domains. Afterwards, the equivalence between coherence and Lawson compactness was generalized to quasicontinuous domains by Lawson [9]. By noticing that every quasicontinuous domain is locally finitary compact and sober [3,4], Jia et al. [6] generalized this equivalence to well-filtered dcpos by throwing away the locally compact property. The key result in [6] is that on well-filtered dcpos L coherence is equivalent to the compactness of $\uparrow x \cap \uparrow y$ for any $x, y \in L$, which seems much weaker than Lawson compactness.

In this paper, we introduce the concept of weak well-filteredness on any space, which will be shown to be strictly weaker than well-filteredness. We prove that the result of Jia et al. is also true on weak well-filtered posets. Meanwhile, we obtain that every weak well-filtered poset is always a consistent dcpo. Moreover, we show that Johnstone's example is weak well-filtered but not well-filtered, in other words, this example illustrates that a weak well-filtered dcpo is not necessarily well-filtered. Finally, at the end of this paper, we show that Lawson compactness is a sufficient condition for a dcpo being coherent.

* Corresponding author.
E-mail address: liqingguoli@aliyun.com (Q. Li).

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2. Preliminaries

We refer to [1,3] for the standard definitions and notations of order theory and domain theory, and to [2,4] for topology.

Let $(L, \leq)$ be a poset. The lower topology, Scott topology and Lawson topology on L are denoted $\omega(L)$, $\sigma(L)$ and $\lambda(L)$, respectively [3]. A topological space is called *well-filtered* if, whenever an open set U contains a filtered intersection $\bigcap_{i\in I} Q_i$ of compact saturated subsets, then U contains Q_i for some $i \in I$. Any sober space is well-filtered (see [3]). We take *coherence* of a topological space to mean that the intersection of any two compact saturated subsets is compact. A *stably compact* space is a topological space which is compact, locally compact, well-filtered and coherent.

Definition 2.1. A topological space (X, τ) is called *weak well-filtered* if, whenever a nonempty open set U contains a filtered intersection $\bigcap_{i\in I} Q_i$ of compact saturated subsets, then U contains Q_i for some $i \in I$.

It is obvious that every well-filtered space is weak well-filtered. The next example illustrates that this implication is strict.

Example 2.1. Consider the poset $L = [0, 1)$ with the usual order of real numbers. $(L, \sigma(L))$ is weak well-filtered. Let $d_n = 1 - 1/n$. Then $\{d_n\}_{n=1}^{\infty}$ is a directed set but it has no a least upper bound in L. Thus $(L, \sigma(L))$ is not well-filtered since every directed set with respect to the specialization order has a least upper bound in a well-filtered T_0-space [8, Proposition 2.4].

We call a poset L weak well-filtered (respectively, well-filtered, compact, sober, coherent, locally compact, stably compact) if L with its Scott topology $\sigma(L)$ is a weak well-filtered (respectively, well-filtered, compact, sober, coherent, locally compact, stably compact) space. Without further reference, we always equip L with the Scott topology $\sigma(L)$. Finally, a poset L is said to be *core-compact* if its Scott topology $\sigma(L)$ is a continuous lattice in the inclusion order.

For a topological space X, we denote the set of all compact saturated sets of X by $\mathcal{Q}(X)$, a canonical topology is generated by the sets

$$\Box U = \{K \in \mathcal{Q}(X) \mid K \subseteq U\},$$

where U ranges over the open subsets of X; this is the so-called upper Vietoris topology. We use $\mathcal{Q}_v(X)$ to denote the resulting topological space. For a poset L, we use $\mathcal{Q}_v(L)$ to denote $\mathcal{Q}_v((L, \sigma(L)))$.

Definition 2.2. [11] A poset L is called a *consistent dcpo* if for any directed subset D of L with $\bigcap_{d\in D} \uparrow d \neq \emptyset$, D has a least upper bound in L.

The specialization order $\leq_\tau$ on a T_0 space (X, τ) is defined by $x \leq_\tau y$ iff $x \in cl_\tau\{y\}$. Let $\uparrow_\tau x = \{y \in X \mid y \leq_\tau x\}$ and $\uparrow_\tau A = \bigcup_{a\in A} \uparrow_\tau a$.

Definition 2.3. [4] Let (X, τ) be a T_0-space. A set $A \subseteq X$ is called *saturated* if it is an intersection of open sets, or equivalently if it is an upper set in the order of specialization. The saturation sat$A$ of a set A is the smallest saturated set containing A; one can easily see that sat$A = \bigcap\{U \in \tau \mid A \subseteq U\}$.

Definition 2.4. [4] A *retract* of a topological space Y is a topological space X such that there are two continuous maps $s : X \to Y$ and $r : Y \to X$ such that $r \circ s = id_X$.