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Weak n-ods, n-ods and strong n-ods

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ABSTRACT

A metric continuum X is:

(a) a weak n-od if there exist subcontinua $A_1, \ldots, A_n$ of X such that $X = A_1 \cup \ldots \cup A_n$, $\emptyset \neq A_1 \cap \ldots \cap A_n$ and no A_i is contained in the union of the others,
(b) an n-od if there exists a subcontinuum A of X such that $X \setminus A$ has at least n components,
(c) a strong n-od if there exists a subcontinuum A of X and nonempty subsets $K_1, \ldots, K_n$ of X such that $X \setminus A = K_1 \cup \ldots \cup K_n$ and $\mathrm{cl}_X(K_i) \cap \mathrm{cl}_X(K_j) = \emptyset$ if $i \neq j$.

In this paper we prove that if X is hereditarily decomposable, then X is an n-od if and only if X is a strong n-od. We also characterize the finite graphs X which are n-ods in terms of its disconnection number and we prove that if X is a finite graph, then X is a weak n-od and it is not an $(n+1)$-od if and only if the dimension of the hyperspace of subcontinua of X is n.

1. Introduction

A *continuum* is a nonempty compact connected metric space. A *mapping* is a continuous function. For an integer $n \geq 2$, a metric continuum X is:

(a) a *weak n-od* if there exist subcontinua $A_1, \ldots, A_n$ of X such that $X = A_1 \cup \ldots \cup A_n$, $\emptyset \neq A_1 \cap \ldots \cap A_n$ and no A_i is contained in the union of the others,
(b) an *n-od* if there exists a subcontinuum A of X such that $X \setminus A$ has at least n components,

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(c) a *strong n-od* if there exists a subcontinuum A of X and nonempty subsets $K_1, \ldots, K_n$ of X such that $X \setminus A = K_1 \cup \ldots \cup K_n$ and $\mathrm{cl}_X(K_i) \cap \mathrm{cl}_X(K_j) = \emptyset$ if $i \neq j$.

A continuum X is *decomposable* if it can be written as the union of two of its proper subcontinua. If X is nondegenerate, X is *indecomposable* if it is not decomposable and X is *hereditarily decomposable* if each of its nondegenerate subcontinua is decomposable.

It is easy to show that being a strong n-od implies being an n-od, and being an n-od implies being a weak n-od (see [8, Proposition 11.25]). Strong 3-ods were introduced in [1] where they were used to study the so called “representation spaces”. In [1, Proposition 4.5] it was shown that for the class of Peano continua, being a 3-od is equivalent to being a strong 3-od. To construct an n-od X such that X is not a strong n-od, we can use the idea of Example 4.4 in [1], namely, take a continuum $X = X_1 \cup \ldots \cup X_n$, with a point p, where $X_1, \ldots, X_{n-2}$ are arcs having p as an end point, X_{n-1} and X_n are indecomposable and $X_i \cap X_j = \{p\}$, if $i \neq j$. The authors of [1] conjectured that this is essentially the only way to construct continua which are n-ods, but no strong n-ods. So, they conjectured [1, Conjecture 4.11] that each such continuum must contain at least two distinct indecomposable subcontinua.

In this paper we prove Conjecture 4.11 of [1] and, as a consequence, we prove that in the class of hereditarily decomposable continua, being an n-od is equivalent to being a strong n-od.

In this paper we are interested in studying continua which are n-ods, weak n-ods or strong n-ods. A general characterization of these type of continua seems to be a very difficult problem. However, for Peano continua we can offer complete characterizations. The authors of [1, Conjecture 4.10] conjectured that there are exactly five Peano continua which are not 3-ods and they are the arc, the simple closed curve, the noose, the dumbell and the theta curve (see descriptions after Corollary 3.9). In fact, they failed to include the figure eight continuum in this list. In this paper we prove that for a Peano continuum X not being an n-od is equivalent to having disconnection number (see Definition 3.5) less than or equal to n (disconnection number was defined by S. B. Nalder, Jr. in [9] and can be computed using the Euler characteristic). And we obtain as a consequence that the six finite graphs mentioned before are the only Peano continua not being 3-ods. In fact, using the results by H. Gladdines, M. van de Vel and B. Vejnar ([3] and [12]), we conclude that there are exactly 26 Peano continua which are not 4-ods.

Given a continuum X, we denote by $C(X)$ the hyperspace of subcontinua of X, endowed with the Hausdorff metric. We prove that if X is a Peano continuum, then X is not a weak n-od if and only if $\dim[C(X)] < n$.

2. Strong n-ods

Let X be a continuum and $x \in X$. We define

$$K(p) = \bigcap\{A \in C(X) : p \in \mathrm{int}(A)\}.$$

We will use the following result proved by H. E. Schlais in [11].

Theorem 2.1 *([11, Theorem 10]). Let X be a continuum. If $K(p)$ has nonempty interior for some $p \in X$, then X is not hereditarily decomposable.*

Corollary 2.2. *Let X be a hereditarily decomposable continuum. Then for each $p \in X$, there exists a proper subcontinuum K of X such that $p \in \mathrm{int}(K)$.*

Proof. Let $p \in X$. If no proper subcontinuum of X contains p in its interior, then $K(p) = X$ and by Theorem 2.1 X is not hereditarily decomposable. A contradiction. $\square$