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Homogeneity degree of some symmetric products

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A B S T R A C T

For a metric continuum X, we consider the nth-symmetric product $F_n(X)$ defined as the hyperspace of all nonempty subsets of X with at most n points. The homogeneity degree $hd(X)$ of a continuum X is the number of orbits for the action of the group of homeomorphisms of X onto itself. In this paper we determine $hd(F_n(X))$ for every manifold without boundary X and $n \in \mathbb{N}$. We also compute $hd(F_n[0,1])$ for all $n \in \mathbb{N}$.

1. Introduction

A *continuum* is a nonempty compact connected metric space.

Here, the word *manifold* refers to a compact connected manifold with or without boundary.

Given a continuum X, the nth-*symmetric product* of X is the hyperspace

$$F_n(X) = \{A \subset X : A \text{ is nonempty and } A \text{ has at most } n \text{ points}\}.$$

The hyperspace $F_n(X)$ is considered with the Vietoris topology.

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Given a continuum X, let $\mathcal{H}(X)$ denote the group of homeomorphisms of X onto itself. An *orbit* in X is a class of the equivalence relation in X given by p is equivalent to q if there exists $h \in \mathcal{H}(X)$ such that $h(p) = q$.

The *homogeneity degree*, $hd(X)$ of the continuum X is defined as

$$hd(X) = \text{ number of orbits in } X.$$

When $hd(X) = n$, the continuum X is known to be $\frac{1}{n}$-homogeneous, and when $hd(X) = 1$, X is *homogeneous*.

In [10], P. Pellicer-Covarrubias studied continua X for which $hd(F_2(X)) = 2$. Recently, I. Calderón, R. Hernández-Gutiérrez and A. Illanes [2] proved that if P is the pseudo-arc, then $hd(F_2(P)) = 3$. Other papers on homogeneity degrees of hyperspaces are [4] and [9].

In this paper we determine $hd(F_n(X))$ for every manifold without boundary X and $n \in \mathbb{N}$. We also compute $hd(F_n[0,1])$ for all $n \in \mathbb{N}$. Since $F_1(X)$ is homeomorphic to X, $hd(F_1(X)) = hd(X)$. Thus, $hd(F_1(X)) = 1$ for every manifold without boundary X and $hd(F_1([0,1])) = 2$.

2. Manifolds without boundary

We denote by S^1 the unit circle in the plane.

Given a continuum X, $n \in \mathbb{N}$ and subsets $U_1, \ldots, U_m$ of X, let

$$\langle U_1, \ldots, U_m \rangle_n = \{A \in F_n(X) : A \subset U_1 \cup \ldots \cup U_m \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, m\}\}.$$

Then the family $\{\langle U_1, \ldots, U_m \rangle_n \subset F_n(X) : m \leq n \text{ and } U_1, \ldots, U_m \text{ are open in } X\}$ is a basis for the Vietoris topology in $F_n(X)$ [7]. If A is any set and $n \in \mathbb{N}$, let $[A]^n = \{B \subset A : |B| = n\}$. Let Y be a topological space. A subset $Z \subset Y$ is a *topological type*, or just a *type*, if for each $h \in \mathcal{H}(Y)$, $h(Z) = Z$.

Lemma 1. *Let X be a locally connected continuum such that $[X]^4$ is a type in $F_4(X)$. Then for each $h \in \mathcal{H}(F_4(X))$, $h([X]^2) \cap [X]^3 = \emptyset$.*

Proof. Suppose to the contrary that there exist $h \in \mathcal{H}(F_4(X))$ and $A = \{a_1, a_2\} \in [X]^2$ such that $h(A) = B = \{b_1, b_2, b_3\} \in [X]^3$.

Let V_1, V_2 and V_3 be pairwise disjoint open connected sets of X such that $b_i \in V_i$ for each $i \in \{1, 2, 3\}$. Let U_1, U_2 be disjoint open connected subsets of X such that $a_1 \in U_1$, $a_2 \in U_2$ and $h(\langle U_1, U_2 \rangle) \subset \langle V_1, V_2, V_3 \rangle$.

The neighborhoods $\langle U_1, U_2 \rangle$ and $\langle V_1, V_2, V_3 \rangle$ have different topological structures. We will use this to arrive to a contradiction.

Let $\mathcal{U} = \langle U_1, U_2 \rangle$ and $\mathcal{V} = h(\mathcal{U})$.

First, we will describe the components of $\mathcal{U} \cap [X]^4$. For each $i \in \{1, 2, 3\}$, let

$$\mathcal{U}_i = \{A \in F_4(X) : |A \cap U_1| = i \text{ and } |A \cap U_2| = 4 - i\}.$$

Then it can be proved that the following properties hold.

(U1) $\mathcal{U}_i$ is a nonempty subset of $\mathcal{U} \cap [X]^4$ for each $i \in \{1, 2, 3\}$,
(U2) $\mathcal{U} \cap [X]^4 = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$,
(U3) $\mathrm{cl}_{F_4(X)}(\mathcal{U}_i) \cap \mathcal{U}_j = \emptyset$ if $i, j \in \{1, 2, 3\}$ and $i \neq j$, and
(U4) $\mathcal{U}_i$ is arcwise connected for all $i \in \{1, 2, 3\}$.

Thus, it follows that $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ are exactly the components of $\mathcal{U} \cap [X]^4$.