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Quasicontinuous and separately continuous functions with values in Maslyuchenko spaces

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We generalize some classical results about quasicontinuous and separately continuous functions with values in metrizable spaces to functions with values in certain generalized metric spaces, called Maslyuchenko spaces. We establish stability properties of the classes of Maslyuchenko spaces and study the relation of these classes to known classes of generalized metric spaces (such as Piotrowski or Stegall spaces). One of our results says that for any $\aleph_0$-space Z and any separately continuous function $f : X \times Y \to Z$ defined on the product of a topological space X and a second-countable space Y, the set $D(f)$ of discontinuity points of f has meager projection on X.

1. Introduction

The problem of evaluation of the sets of discontinuity points of quasicontinuous and separately continuous functions is classical in Real Analysis and traces its history back to the famous dissertation of Baire [1].

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Let us recall that a function $f : X \to Y$ between topological spaces is *quasicontinuous at a point* $x \in X$ if for any neighborhood $O_x \subset X$ of x and any neighborhood $O_{f(x)} \subset Y$ of $f(x)$ there exists a non-empty open set $U \subset O_x$ such that $f(U) \subset O_{f(x)}$. A function $f : X \to Y$ is *quasicontinuous* if it is quasicontinuous at each point $x \in X$. Formally, quasicontinuous functions were introduced by Kempisty [13] but implicitly they appeared earlier in works of Baire and Volterra.

The following property of quasicontinuous functions is well-known (and can be easily derived from the definition), see [5], [17], [31], [32], [33].

Theorem 1.1. *For any quasicontinuous function $f : X \to Y$ from a topological space X to a metrizable space Y the set $D(f)$ of discontinuity points of f is meager in X.*

We recall that a subset M of a topological space X is *meager* in X if M can be written as the countable union of nowhere dense subsets of X.

In fact, the metrizability of the space Y in Theorem 1.1 can be weakened to the strict fragmentability of Y. We recall that a topological space Y is *fragmented* by a metric d if for every $\varepsilon > 0$, each non-empty subspace $A \subset X$ contains a non-empty relatively open subset $U \subset A$ of d-diameter $\mathrm{diam}(U) < \varepsilon$. If the metric d generates a topology at least as strong as the original topology of X, then we shall say that X is *strictly fragmented* by the metric d. A topological space is called (*strictly*) *fragmentable* if it is (strictly) fragmented by some metric. It is clear that each metrizable space is strictly fragmentable and each strictly fragmentable space is fragmentable. By [35], a compact Hausdorff space is strictly fragmentable if and only if it is fragmentable. In [14] this equivalence was generalized to so-called game determined spaces. The following generalization of Theorem 1.1 was obtained in [9, Theorem 2.7], [29, Lemma 4.1], [15, Theorem 5.1], [14, Theorem 1].

Theorem 1.2 *(Giles–Kenderov–Kortezov–Moors–Sciffer). For any quasicontinuous function $f : X \to Y$ from a topological space X to a strictly fragmentable space Y the set $D(f)$ of discontinuity points of f is meager in X.*

In [34] Piotrowski suggested to study spaces Y for which every quasicontinuous function $f : X \to Y$ defined on a non-empty Baire space X has a continuity point. Topological spaces Y possessing this property are called *Piotrowski* spaces. Properties of Piotrowski spaces are discussed in the survey paper [3]. By [3], a regular topological space X is Piotrowski if and only if for any quasicontinuous function $f : Z \to X$ the set $D(f)$ is discontinuity points of f is meager in Z.

Concerning discontinuity points of separately continuous functions we have the following classical result due to Calbrix and Troallic [7].

Theorem 1.3 *(Calbrix–Troallic). Let X be a topological space, Y be a second countable space and Z be a metrizable space. For any separately continuous function $f : X \times Y \to Z$ the set $D(f)$ of discontinuity points of f has meager projection on X.*

In [19] this theorem was generalized to KC-functions. Let X, Y, Z be topological spaces. A function $f : X \times Y \to Z$ is called a *KC-function* if

- for every $y \in Y$ the function $f_y : X \to Z$, $f_y : x \mapsto f(x, y)$, is quasicontinuous, and
- for every $x \in X$ the function $f^x : Y \to Z$, $f^x : y \mapsto f(x, y)$, is continuous.