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On pseudocompact spaces with a weak selection

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A B S T R A C T

We study properties of the pseudocompact spaces X with a weak selection, and we dedicate a particular attention to the weak selection topologies on X. In case when X is also locally compact, we obtain a convenient decomposition of X into a finite union of clopen sets, which are either almost compact or connected with a remainder of size two in their Stone–Čech compactification.

1. Introduction

1.1. Background on selections

All spaces in this paper are assumed to be Tychonoff. Let X be a space and $\mathcal{F}_2(X) = \{S \subset X : |S| \leq 2\}$ endowed with the *Vietoris topology* τ_V which is generated by all collections of the following form:

$$\langle \mathcal{V} \rangle = \{S \in \mathcal{F}_2(X) : S \subseteq \bigcup \mathcal{V}, S \cap V \neq \emptyset, \text{ for } V \in \mathcal{V}\},$$

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where $\mathcal{V}$ runs over the finite families of open subsets of X.

A map $\sigma : \mathcal{F}_2(X) \to X$ is called a *weak selection* for X if $\sigma(S) \in S$ for every $S \in \mathcal{F}_2(X)$. We say that $\sigma : \mathcal{F}_2(X) \to X$ is *continuous* if it is continuous with respect to the Vietoris topology τ_V on $\mathcal{F}_2(X)$. We use $\mathcal{S}\mathrm{el}_2(X)$ to denote the set of all continuous weak selections on X.

Let X be a space with $\mathcal{S}\mathrm{el}_2(X) \neq \emptyset$ and $\sigma \in \mathcal{S}\mathrm{el}_2(X)$. For $x, y \in X$ with $x \neq y$ we write $x \prec_\sigma y$ if $\sigma(\{x, y\}) = x$, and let

$$(\leftarrow, x)_\sigma = \{y \in X : y \prec_\sigma x\} \text{ and } (x, \rightarrow)_\sigma = \{y \in X : x \prec_\sigma y\}.$$

The *selection topology* τ_σ on X is generated by the family

$$\{(\leftarrow, x)_\sigma, (x, \rightarrow)_\sigma : x \in X\}$$

as a subbase [13]. It is always Tychonoff [16] and coarser than the original topology τ of X, but it need not be normal [11]. By $\mathcal{T}_{sel}(X)$ we denote the set $\{\tau_\sigma : \sigma \in \mathcal{S}\mathrm{el}_2(X)\}$ of all selection topologies on X for continuous weak selections.

In this paper, every order is assumed to be total. We say that a space X is:

(a) *orderable* if there exists an order on X generating the topology;
(b) *suborderable* if X can be embedded in an orderable space;
(c) *weakly orderable* if there exists a coarser orderable topology on X.

In these terms it is well-known that one has:

$$\text{orderable} \overset{(i)}{\Longrightarrow} \text{suborderable} \overset{(ii)}{\Longrightarrow} \text{weakly orderable} \overset{(iii)}{\Longrightarrow} \mathcal{S}\mathrm{el}_2(X) \neq \emptyset \tag{1}$$

The non-reversibility of the implication (i) is witnessed by the Sorgenfrey line. The non-reversibility of (ii) is witnessed by the sequential fan S_ω defined by Arhangel'skij and Franklin. Answering van Mill–Wattel's problem [20], Hrušák and Martinez–Ruiz [15] proved the non-reversibility of (iii).

On the other hand, van Mill and Wattel [20] proved that all implications (i)–(iii) in (1) become reversible for compact spaces:

Theorem 1.1 *([20]). A compact space X is orderable iff $\mathcal{S}\mathrm{el}_2(X) \neq \emptyset$.*

This remarkable theorem motivated the study of the impact of compact like properties on the reversibility of the implications in (1). The case when "compact" is replaced by "pseudocompact" seems to be the most understood one. As we see in the sequel, here the implications (ii) and (iii) in (1) are reversible (see Theorem 1.2), while (i) is reversible if one imposes almost compactness on the space (see Theorem 1.4 and Example 5.1).

Purisch [23] proved that βX is orderable if and only if X is pseudocompact and suborderable, essentially using the fact that if βX is orderable, then X is normal and countably compact, established by Venkataraman, Rajagopalan and Soundararajan [25]. van Douwen [7] proved that for a space X with $\mathcal{S}\mathrm{el}_2(X) \neq \emptyset$ countable compactness is equivalent to sequential compactness. The equivalence of (a) and (b) in Theorem 1.2 was proved by Artico–Marconi–Pelant–Rotter–Tkachenko [1] and independently by Miyazaki [21]. The question whether every pseudocompact space X with $\mathcal{S}\mathrm{el}_2(X) \neq \emptyset$ is suborderable was raised in [1]. It was positively answered by Garcia-Ferreira and Sanchis [9]. Combining all these results one gets: