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Selectively sequentially pseudocompact group topologies on torsion and torsion-free Abelian groups

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A B S T R A C T

A space X is *selectively sequentially pseudocompact* if for every family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence. Let G be a group from one of the following three classes: (i) $\mathcal{V}$-free groups, where $\mathcal{V}$ is an arbitrary variety of Abelian groups; (ii) torsion Abelian groups; (iii) torsion-free Abelian groups. Under the Singular Cardinal Hypothesis SCH, we prove that if G admits a pseudocompact group topology, then it can also be equipped with a selectively sequentially pseudocompact group topology. Since selectively sequentially pseudocompact spaces are strongly pseudocompact in the sense of García-Ferreira and Ortiz-Castillo, this provides a strong positive (albeit partial) answer to a question of García-Ferreira and Tomita.

The symbol $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ denotes the set of positive integer numbers. The symbol $\mathbb{Z}$ denotes the group of integer numbers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{T}$ denotes the circle group $\{e^{i\theta} : \theta \in \mathbb{R}\} \subseteq \mathbb{R}^2$. The symbols ω, ω_1 and $\mathfrak{c}$ stand for the first infinite cardinal, the first uncountable cardinal and the cardinality of the continuum, respectively. For a set X, the set of all finite subsets of X is denoted by $[X]^{<\omega}$, while $[X]^\omega$ denotes the set of all countably infinite subsets of X.

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If X is a subset of a group G, then $\langle X\rangle$ is the smallest subgroup of G that contains X.

For groups which are not necessary Abelian we use the multiplication notation, while for Abelian groups we always use the additive one. In particular, e denotes the identity element of a group and 0 is used for the zero element of an Abelian group. Recall that an element g of a group G is *torsion* if $g^n = e$ for some positive integer n. A group is *torsion* if all of its elements are torsion. A group G is *bounded torsion* if there exists $n \in \mathbb{N}$ such that $g^n = e$ for all $g \in G$.

Let G be an Abelian group. The symbol $t(G)$ stands for the subgroup of torsion elements of G. For each $n \in \mathbb{N}$, note that $nG = \{ng : g \in G\}$ is a subgroup of G and the map $g \mapsto ng$ $(g \in G)$ is a homomorphism of G onto nG. For a cardinal τ we denote by $G^{(\tau)}$ the direct sum of τ copies of the group G.

When groups G and H are isomorphic, we write $G \cong H$.

We will say that a sequence $\{x_n : n \in \mathbb{N}\}$ of points in a topological space X:

- *converges to a point* $x \in X$ if the set $\{n \in \mathbb{N} : x_n \notin W\}$ is finite for every open neighbourhood W of x in X;
- is *convergent* if it converges to some point of X.

1. Introduction

In this paper, we continue the study of the class of selectively sequentially pseudocompact spaces introduced by the authors in [9].

Definition 1.1. [9] A topological space X is *selectively sequentially pseudocompact* if for every sequence $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence.

A nice feature of the class of selectively sequentially pseudocompact spaces is that it is closed under taking arbitrary Cartesian products:

Proposition 1.2. [9, Corollary 4.4] *Arbitrary products of selectively sequentially pseudocompact spaces are selectively sequentially pseudocompact.*

García-Ferreira and Ortiz-Castillo [12] have recently introduced the class of strongly pseudocompact spaces. The original definition involves the notion of a p-limit for an ultrafilter p on $\mathbb{N}$. The authors have shown in [9, Theorem 2.1] that the original definition is equivalent to the following one:

Definition 1.3. A topological space X is called *strongly pseudocompact* provided that for each sequence $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint non-empty open subsets of X, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ such that the set $\{x_n : n \in \mathbb{N}\}$ is not closed in X.

In the class of topological groups, this property has been investigated by García-Ferreira and Tomita in [13].

The authors proposed to call strongly pseudocompact spaces *selectively pseudocompact* in [9, Definition 2.2], and this terminology was later adopted in [14] and [17]. Nevertheless, in this paper we continue to use the original name "strongly pseudocompact" for the class of spaces described in Definition 1.3.

The following implications are clear from Definitions 1.1 and 1.3:

$$\text{selectively sequentially pseudocompact} \to \text{strongly pseudocompact} \to \text{pseudocompact}. \tag{1}$$