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Gauss diagrams, unknotting numbers and trivializing numbers of spatial graphs

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In this paper we introduce Gauss diagrams and four kinds of unknotting numbers of a spatial graph. R. Hanaki introduced the notion of pseudo diagrams and the trivializing numbers of knots, links and spatial graphs whose underlying graphs are planar. We generalize the trivializing numbers without the assumption that the underlying graphs are planar. Finally we discuss relations among the unknotting numbers and the trivializing numbers.

1. Introduction

Spatial graphs are graphs embedded in 3-dimensional Euclidean space $\mathbb{R}^3$. As knots and links are encoded by Gauss diagrams with chords on circles, it is natural to extend similar ideas to encode spatial graphs using Gauss diagrams. In this paper, we first discuss a way to represent spatial graph diagrams using Gauss diagrams. In [1], [2], [3], T. Fleming and B. Mellor generalized the concept of Gauss codes for virtual spatial graphs. It seems important to encode spatial graphs using Gauss diagrams or Gauss codes when we treat them by computer, and it would be expected that many invariants of spatial graphs can be computed in terms of Gauss diagrams.

We introduce Gauss diagrams for spatial graphs in Section 2. The construction is similar to that of Gauss diagrams for knots. Some typical examples of the moves on Gauss diagrams corresponding to Reidemeister moves on spatial graph diagrams are shown. Some moves, RII moves and RIV moves, may change Gauss

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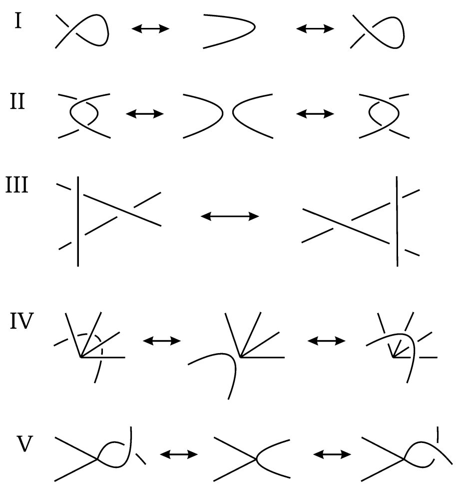

Fig. 1. Reidemeister moves for spatial graph diagrams.

diagrams of spatial graph diagrams to Gauss diagrams that do not correspond to spatial graph diagrams. We discuss this in Section 3. In Section 4, we discuss Gauss diagrams for based spatial graphs.

In Section 5, we introduce four kinds of unknotting numbers of a spatial graph: the unknotting number, the based unknotting number, the Γ-unknotting number and the based Γ-unknotting number. This is based on the idea discussed in [8]. R. Hanaki [5] introduced the notion of pseudo diagrams and the trivializing numbers for knots, links, and spatial graphs, where he assumed that the underlying graphs of spatial graphs are planar. A. Henrich et al. [6] gave a method of computing the trivializing number of a regular projection of a knot using Gauss diagram. In Section 6, we define four kinds of trivializing numbers: the trivializing number, the based trivializing number, the Γ-trivializing number, and the based Γ-trivializing number. In Section 7, we give inequalities among the unknotting numbers and the trivializing numbers.

2. Spatial graphs and Gauss diagrams

A *spatial graph* is a finite graph in $\mathbb{R}^3$. Two spatial graphs are said to be *equivalent* if they are ambiently isotopic. For simplicity, throughout this paper, we assume that a spatial graph is connected, there are no degree-0 vertices and no degree-1 vertices, and that there is at least one vertex whose degree is greater than two. Furthermore, we assume that a spatial graph is oriented, i.e., the edges are oriented.

Similar to a diagram of a knot, a regular projection of a spatial graph G is obtained by projecting G to a plane so that the multiple points are transverse double points away from vertices. A diagram of G is a regular projection of G in $\mathbb{R}^2$ with over/under information at each double point. A double point with over/under information is called a *crossing*. L.H. Kauffman [7] and D.N. Yetter [9] proved that two diagrams present equivalent spatial graphs if and only if they are related by the moves shown in Fig. 1. We refer to these moves as Reidemeister moves for spatial graph diagrams.

A *Gauss diagram* of a knot diagram is an oriented circle identified with the source circle of an embedding into $\mathbb{R}^3$ whose image is the knot, and some chords attached to the circle whose endpoints correspond to over crossings and under crossings of the crossings. The chords are oriented from over crossings to under crossings. When a crossing of the diagram is denoted by c, then the endpoints of the corresponding chord will be denoted by $\overline{c}$ and $\underline{c}$, where $\overline{c}$ is the over crossing and $\underline{c}$ is the under crossing. Chords are assigned signs which are equal to the signs of the crossings. See Fig. 2.

Reidemeister moves for Gauss diagrams are shown in Fig. 3. When we apply moves of type II to Gauss diagrams of knots, we may obtain Gauss diagram that do not correspond to knot diagrams. Such Gauss