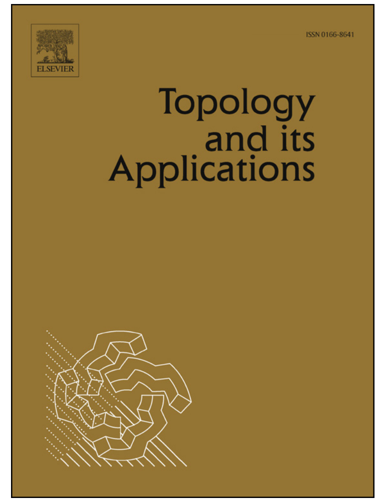
Accepted Manuscript



Quantale-valued Topological Spaces via Closure and Convergence

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PII: S0166-8641(17)30422-4
DOI: http://dx.doi.org/10.1016/j.topol.2017.08.038
Reference: TOPOL 6226

To appear in: *Topology and its Applications*

Received date: 30 April 2016
Revised date: 29 September 2016
Accepted date: 30 September 2016

Please cite this article in press as: H. Lai, W. Tholen, Quantale-valued Topological Spaces via Closure and Convergence, *Topol. Appl.* (2017), http://dx.doi.org/10.1016/j.topol.2017.08.038

Quantale-valued Topological Spaces via Closure and Convergence

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Abstract

For a quantale V we introduce V-valued topological spaces via V-valued point-set-distance functions and, when V is completely distributive, characterize them in terms of both, so-called closure towers and ultrafilter convergence relations. When V is the two-element chain 2, the extended real half-line $[0, \infty]$, or the quantale $\mathbf{\Delta}$ of distance distribution functions, the general setting produces known and new results on topological spaces, approach spaces, and the only recently considered probabilistic approach spaces, as well as on their functorial interactions with each other.

Keywords: quantale, V-valued closure space, V-valued topological space, discrete V-presheaf monad, lax distributive law, lax (λ, V)-algebra, probabilistic approach space, algebraic functor, change-of-base functor.
2010 MSC: 54A20, 54B30, 54E70, 18D20, 18C99.

1. Introduction

Lowen's [17] approach spaces provide an ideal synthesis of Lawvere's [16] presentation of metric spaces (as small $[0, \infty]$-enriched categories) and the Manes-Barr [19, 1] presentation of topological spaces in terms of ultrafilter convergence, as demonstrated first in [4]; see also [12]. Several authors have investigated *probabilistic* generalizations of these concepts (see in particular [20, 3, 11, 14]), which suggests that a quantale-based study of generalized topological spaces should be developed, in order to treat these and other new concepts efficiently in a unified manner, in terms of both, "distance" or "closure", and "convergence". In this paper we provide such a treatment, working with an arbitrary quantale $\mathsf{V} = (\mathsf{V}, \otimes, \mathsf{k})$ which, for the main results of the paper, is required to be completely distributive. For $\mathsf{V} = 2$ the two-element chain, our results reproduce the equivalence of the descriptions of topologies in terms of closure and ultrafilter convergence; for $\mathsf{V} = [0, \infty]$ (ordered by the natural $\geq$ and structured by $+$ as the quantalic $\otimes$), one obtains the known equivalent descriptions of approach spaces in terms of point-set distances and of ultrafilter convergence; for $\mathsf{V} = \mathbf{\Delta}$ the quantale of *distance distribution functions* $\varphi : [0, \infty] \longrightarrow [0, 1]$, required to satisfy the left-continuity condition $\varphi(\beta) = \sup_{\alpha<\beta}\varphi(\alpha)$ for all $\beta \in [0, \infty]$, the corresponding equivalence is established here also for *probabilistic approach spaces*. A major advantage of working in the harmonized context of a general quantale is that it actually makes the proofs more transparent to us than if they were carried out in the concrete quantales that we are interested in.

While this paper is built on the methods of *monoidal topology* as developed in [6, 5, 12] and elsewhere (see in particular [13]), in this paper we emphasize the lax-algebraic setting presented in [22], which is summarized in this paper to the extent needed. This setting is in fact well motivated by Lowen's original axioms for an approach space (X, δ) in terms of its point-set distance function $\delta : X \times \mathsf{P}X \longrightarrow [0, \infty]$, listed in [17] with $\mathsf{P}X = 2^X$, as follows:

(D1) $\forall x \in X : \ \delta(x, \{x\}) = 0$,
(D2) $\forall x \in X : \ \delta(x, \emptyset) = \infty$,

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[1] Partial financial assistance by National Natural Science Foundation of China (11101297), International Visiting Program for Excellent Young Scholars of Sichuan University, and by the Natural Sciences and Engineering Research Council (NSERC) of Canada is gratefully acknowledged. This work was completed while the first author held a Visiting Professorship at York University.