Topology and its Applications 230 (2017) 621–638

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – Proceedings of the First Pan-Pacific International Conference on Topology and Applications (PPICTA)

Genera and minors of multibranched surfaces

Shosaku Matsuzaki [a], Makoto Ozawa [b,*]

[a] *Faculty of Engineering, Takushoku University, Tatemachi 815-1, Hachioji-shi, Tokyo, 193-0985, Japan*
[b] *Department of Natural Sciences, Faculty of Arts and Sciences, Komazawa University, 1-23-1 Komazawa, Setagaya-ku, Tokyo, 154-8525, Japan*

ARTICLE INFO

Article history:
Received 31 March 2016
Received in revised form 22 September 2016
Accepted 22 September 2016
Available online 24 August 2017

MSC:
57Q35
57N35

Keywords:
CW complex
Multibranched surface
Genus
Minor
Heegaard genus
Intrinsically knotted
Intrinsically linked
Obstruction set

ABSTRACT

We say that a 2-dimensional CW complex is a *multibranched surface* if we remove all points whose open neighborhoods are homeomorphic to the 2-dimensional Euclidean space $\mathbb{R}^2$, then we obtain a 1-dimensional complex which is homeomorphic to a disjoint union of some S^1's. We define the genus of a multibranched surface X as the minimum number of genera of 3-dimensional manifold into which X can be embedded. We prove some inequalities which give upper bounds for the genus of a multibranched surface. A multibranched surface is a generalization of graphs. Therefore, we can define "minors" of multibranched surfaces analogously. We study various properties of the minors of multibranched surfaces.

1. Introduction

It is a fundamental problem to determine whether or not there exists an embedding from a topological space into another one. The Menger–Nöbeling theorem ([4, Theorem 1.11.4.]) shows that any finite 2-dimensional CW complex can be embedded into the 5-dimensional Euclidian space $\mathbb{R}^5$. This is a best possible result since for example, the union of all 2-faces of a 6-simplex cannot be embedded in $\mathbb{R}^4$ ([4, 1.11.F]). But, if the subspace of a finite 2-dimensional CW complex consisting of all points which do not have an open neighborhood homeomorphic to $\mathbb{R}^2$ is a possibly disconnected 1-dimensional manifold, then the CW complex can be embedded into $\mathbb{R}^4$ (Proposition 2.3). We call such CW complexes *multibranched surfaces*, and moreover obtain a necessary and sufficient condition for a multibranched surface to be embeddable

* Corresponding author.
E-mail addresses: smatsuza@ner.takushoku-u.ac.jp (S. Matsuzaki), w3c@komazawa-u.ac.jp (M. Ozawa).

http://dx.doi.org/10.1016/j.topol.2017.08.041

into some closed orientable 3-dimensional manifold (Proposition 2.7). This is a starting point to study such multibranched surfaces via 3-dimensional manifolds.

In this paper, we introduce the *genus* of such a multibranched surface as the minimal Heegaard genus of 3-dimensional manifolds into which the multibranched surface can be embedded. In section 3, we will give some upper bounds for the genus of a multibranched surface (Theorems 3.5, 3.6). In section 4, we will describe the first homology groups of multibranched surfaces, and calculate for some examples. It can be used to determine whether or not a multibranched surface can be embedded into the 3-sphere. This constructively explains more details of the calculation by using the determinant of some matrix in [5]. In [5], we also studied the criticality of a multibranched surface for the 3-sphere S^3 and have given some critical multibranched surfaces for S^3. As the Kuratowski's theorem ([11]) characterized the 2-sphere S^2 by means of the obstruction set of graphs, it might be possible to characterize a closed 3-dimensional manifold by means of some obstruction set of multibranched surfaces.

In Graph Theory, Robertson and Seymour introduced the minor theory which gives a most important structure on the set of graphs. Since we can regard a graph as a 1-dimensional multibranched manifold, it would be natural to consider a similar minor theory for multibranched surfaces. In Section 5, we will define the minor for multibranched surfaces and introduce some intrinsic properties which are minor closed. Thus we arrive at the obstruction set for those intrinsic properties, and give some examples which belong to the obstruction set. And also we define the neighborhood minor for multibranched surfaces. The neighborhood minor sets a preorder on the set of multibranched surfaces, and behaves well on some basic operations (Proposition 5.14). In particular, if X is a neighborhood minor of Y, then the genus of X is less than or equal to that of Y.

To summarize this paper, we have found a well-behaved class of 2-dimensional CW complexes (which we call *regular multibranched surfaces*) and derived an invariant of regular multibranched surfaces from the Heegaard genus of 3-dimensional manifolds which is known to be the most fundamental invariant of 3-dimensional manifolds. In the future, we expect some characterization of each 3-dimensional manifold by means of the obstruction set of regular multibranched surfaces like as the Kuratowski's theorem.

2. Preliminaries

2.1. Multibranched surfaces

Let $\mathbb{R}^n_+$ be the upper half space $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n | x_n \geq 0\}$ of n-dimensional Euclidean space $\mathbb{R}^n$. The quotient space obtained from i copies of $\mathbb{R}^n_+$ by identifying with their boundaries, $\partial\mathbb{R}^n_+ = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n | x_n = 0\}$, is denoted by S^n_i. We note that S^n_2 is homeomorphic to $\mathbb{R}^n$. See Fig. 1.

Definition 2.1. An n-dimensional CW complex X is an *n-dimensional multibranched manifold* if for every point $x \in X$ there exist a positive integer i and an open neighborhood U of x such that U is homeomorphic to S^n_i.

We call a 2-dimensional multibranched manifold a *multibranched surface*. In this paper, we consider multibranched surfaces which are constructed by gluing some compact 2-dimensional manifolds into their boundaries.

We prepare a closed 1-dimensional manifold L, a compact 2-dimensional manifold E and a continuous map $\phi : \partial E \to L$ satisfying the following conditions.

1. For every connected component e of E, $\partial e \neq \emptyset$.
2. For every connected component c of ∂E, the restriction $\phi|_c : c \to \phi(c)$ is a covering map.