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Mahavier completeness and classifying diagrams

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A B S T R A C T

Generalised inverse limits of compacta were introduced by Ingram and Mahavier in 2006. The main difference between ordinary inverse limits and their generalised cousins is that the former concerns diagrams of singlevalued functions while the latter permits multivalued functions. However, generalised inverse limits are not merely limits in the Kleisli category of a hyperspace monad, a fact that independently motivated each of the authors of this article to come up with the same formalism which restores the link with category theory through the concept of Mahavier limit of an order diagram in an order extension of a category $\mathscr{B}$. Mahavier limits of diagrams in $\mathscr{B}$ coincide with ordinary limits in $\mathscr{B}$, and so Mahavier limits are an extension of ordinary limits along the functor that views an ordinary diagram as a diagram in the extension. Within that context it is natural to consider Mahavier completeness, namely when all small diagrams admit Mahavier limits, as well as classifying diagrams, namely the existence of a right adjoint to the mentioned functor on diagrams. In this work we show that these two conditions are equivalent, and we study some of the properties of classifying diagrams and of the adjunction.

1. Introduction

Generalised inverse limits of compacta were introduced by Ingram and Mahavier in 2006 in [1] and have since received much attention (e.g., [2–24]). Recall that an inverse limit of a sequence

$$\cdots \longrightarrow X_{n+1} \xrightarrow{f_n} X_n \xrightarrow{f_{n-1}} \cdots \longrightarrow X_2 \xrightarrow{f_1} X_1$$

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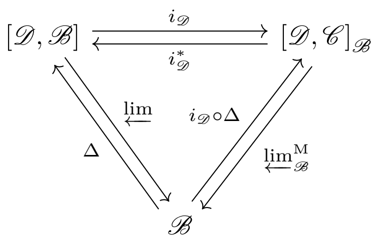
of spaces and continuous functions is the space $X = \{\boldsymbol{x} \in \prod X_n \mid \boldsymbol{x}_n = f_n(\boldsymbol{x}_{n+1})\}$, viewed as a subspace of the product space. The passage to generalised inverse limits occurs by allowing the bonding functions $f_n\colon X_{n+1} \to X_n$ to be upper semicontinuous set-valued functions $f_n\colon X_{n+1} \rightsquigarrow X_n$, and by altering the definition of the space X to become $X = \{\boldsymbol{x} \in \prod X_n \mid \boldsymbol{x}_n \in f_n(\boldsymbol{x}_{n+1})\}$. The formal resemblance to inverse limits makes the generalised version very palatable. The hoard of interesting spaces that arise as generalised inverse limits of very simple diagrams with multivalued bonding functions of compacta (see [11,25] for detailed examples), together with highly non-trivial ramification of the subtle change in definition from singlevaluedness to multivaluedness, and from equality to membership, contribute even more to the appeal of this relatively new area of research.

Of course, inverse limits of spaces are nothing but categorical limits in the category **Top** of topological spaces and continuous mappings, and it is natural to ask whether the slogan generalises. Results addressing some categorical aspects of generalised inverse limits directly can be found in [4,26], but they were only partially successful in fully restoring the link with category theory, and the difficulty can be traced to the following phenomenon. Consider the functor $T\colon \mathbf{Top} \to \mathbf{Top}$ which maps a space X to $T(X)$, the space of all subsets of X, endowed with the upper Vietoris topology. This hyperspace functor has a natural structure of a monad whose multiplication is given by taking unions. Let $\mathbf{Top}_T$ be the Kleisli category of T, i.e., the objects of $\mathbf{Top}_T$ are all spaces and a morphism $X \rightsquigarrow Y$ is a continuous function $X \to T(Y)$. It is easily seen that these are precisely the upper semicontinuous functions. In other words, the diagrams for generalised inverse limits of spaces are precisely diagrams in $\mathbf{Top}_T$. However, generalised inverse limits in **Top** are not simply limits in $\mathbf{Top}_T$ (an expected reality since limits in Kleisli categories are notoriously ill-behaved ([27]), while generalised inverse limits are much more tame).

The authors of this article independently found the same categorical formalism to fully restore the link between generalised inverse limits of spaces and category theory. In [18] the first named author developed a notion of limit in the category of compacta and upper semicontinuous set-valued functions in such a way that the slogan above is recovered. In [28] the second named author developed a formalism in full generality, allowing for generalised inverse limits to be considered beyond the scope of topology, which specialises to generalised inverse limits of spaces when interpreted in the context of $\mathbf{Top} \subseteq \mathbf{Top}_T$.

The aim of this work is summarised in the diagram



which we briefly explain (all concepts are detailed below). Let $\mathscr{B}, \mathscr{C}$, and $\mathscr{D}$ be categories (with $\mathscr{D}$ small), assume that $\mathscr{B}$ is a subcategory of $\mathscr{C}$, that $\mathrm{ob}(\mathscr{B}) = \mathrm{ob}(\mathscr{C})$, and moreover that each hom-set in $\mathscr{C}$ is endowed with an ordering, with some conditions. We call $\mathscr{C}$ an *order extension* of $\mathscr{B}$. The ordering allows one to define order variants of functors and of natural transformations by suitably replacing $=$ by $\leq$. One obtains in this way the category $[\mathscr{D}, \mathscr{C}]$ of all order functors $\mathscr{D} \to \mathscr{C}$ and order natural transformations between them. An order natural transformation whose components are morphisms in $\mathscr{B}$ is said to be an *order natural transformation relative* to $\mathscr{B}$, and we then denote by $[\mathscr{D}, \mathscr{C}]_{\mathscr{B}}$ the subcategory of $[\mathscr{D}, \mathscr{C}]$ obtained by restricting to the relative order natural transformations. Let $[\mathscr{D}, \mathscr{B}]$ be the usual category of functors $\mathscr{D} \to \mathscr{B}$ and natural transformations. Since $\mathscr{B}$ is a subcategory of $\mathscr{C}$ there is an inclusion functor $i_{\mathscr{D}}\colon [\mathscr{D}, \mathscr{B}] \to [\mathscr{D}, \mathscr{C}]_{\mathscr{B}}$, depicted at the top of the diagram above. On the left side of the diagram are the diagonal functor $\Delta\colon \mathscr{B} \to [\mathscr{D}, \mathscr{B}]$, mapping an object B to the constantly B functor, and its right adjoint, the functor $\varprojlim$, namely taking limits, provided $\mathscr{D}$-shaped limits in $\mathscr{B}$ exist, e.g., if $\mathscr{B}$ is complete.