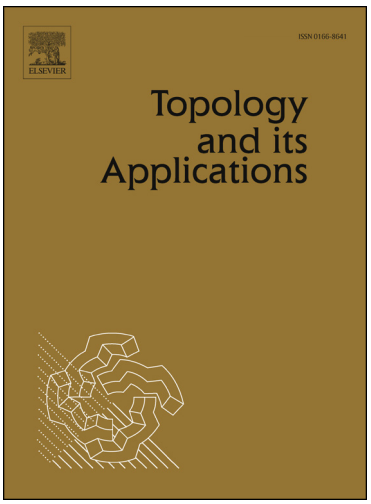

Incompressible solvable representations of surface groups

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PII: S0166-8641(17)30352-8
DOI: http://dx.doi.org/10.1016/j.topol.2017.07.009
Reference: TOPOL 6180

To appear in: *Topology and its Applications*

Received date: 25 May 2017
Accepted date: 18 July 2017

Please cite this article in press as: J. DeBlois, D. Gomez, Incompressible solvable representations of surface groups, *Topol. Appl.* (2017), http://dx.doi.org/10.1016/j.topol.2017.07.009

INCOMPRESSIBLE SOLVABLE REPRESENTATIONS OF SURFACE GROUPS

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Abstract. The fundamental group of every surface that is not the projective plane or Klein bottle has a representation to a torsion-free group of upper-triangular matrices in $\mathrm{SL}_2(\mathbb{R})$ with no simple loop (i.e. a nontrivial element representing a simple closed curve) in the kernel.

We will call a representation (i.e. a homomorphism) from the fundamental group $\pi_1 S$ of a surface S to another group *incompressible* if its kernel contains no simple loop; i.e. a non-trivial element representing a simple closed curve in S.

Theorem. *For any compact surface Σ, possibly with boundary, that is not the projective plane or Klein bottle there is an incompressible representation of $\pi_1\Sigma$ to a torsion-free group of upper triangular matrices in $\mathrm{SL}_2(\mathbb{R})$, which therefore injects under the quotient map to $\mathrm{PSL}_2(\mathbb{K})$ or $\mathrm{PGL}_2(\mathbb{K})$, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*

Remark 1. It is not hard to show directly that the projective plane's and Klein bottle's fundamental groups have incompressible representations to upper-triangular matrices in $\mathrm{SL}_2(\mathbb{R})$, but these all have torsion and do not remain incompressible after projectivizing. See Remark 3.

Remark 2. The theorem above implies in particular that every simple loop in $\pi_1\Sigma$ survives its *metabelianization*, the quotient by the second term of its derived series. As a sort of sanity check, we note that this was shown directly in [13, Example 18].

This implies an(other) answer to a question of Yair Minsky. Motivated by the three-dimensional simple loop conjecture, he asked whether there exist non-injective but incompressible representations of hyperbolic surface groups into $\mathrm{PSL}_2(\mathbb{C})$ [10, Question 5.3]. Minsky's question has been answered in the affirmative, rather emphatically at this point, in independent works of Cooper–Manning [4], Louder [7], Danny Calegari [3], and Mann [8]. These interesting papers each provide non-injective, incompressible representations that are customized in different ways.

We think this result is worth adding to the pile for a few reasons. First, our representations are as simple as possible, both in terms of the target's dimension (two) and the fact that it is two-step solvable — any one-step solvable (i.e. abelian) representation kills all nullhomologous curves. And their existence is not implied by previous work. The set of upper triangular representations in $\mathrm{SL}_2(\mathbb{R})$ has positive codimension in the variety considered by Cooper–Manning; Louder's representations factor through embeddings of limit groups; Mann's are faithful on certain free subgroups of $\pi_1\Sigma$; and Calegari certifies incompressibility using positivity of *scl*, which vanishes on the commutator subgroup of a solvable group (cf. [2]).

Our proof also represents a particular approach to proving existence of representations with nice properties, boiled down to its bare bones. We show for any fixed simple closed curve that the set of upper-triangular representations that kill