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Cancellation for 4-manifolds with virtually abelian fundamental group

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ABSTRACT

Suppose X and Y are compact connected topological 4-manifolds with fundamental group π. For any $r \geqslant 0$, X **is** r**-stably homeomorphic to** Y if $X \# r(S^2 \times S^2)$ is homeomorphic to $Y \# r(S^2 \times S^2)$. How close is stable homeomorphism to homeomorphism?

When the common fundamental group π is virtually abelian, we show that large r can be diminished to $n+2$, where π has a finite-index subgroup that is free-abelian of rank n. In particular, if π is finite then $n = 0$, hence X and Y are 2-stably homeomorphic, which is one $S^2 \times S^2$ summand in excess of the cancellation theorem of Hambleton–Kreck [12].

The last section is a case study of the homeomorphism classification of closed manifolds in the tangential homotopy type of $X = X_- \# X_+$, where $X_\pm$ are closed nonorientable topological 4-manifolds with order-two fundamental groups [13].

1. Introduction

Suppose X is a compact connected smooth 4-manifold, with fundamental group π and orientation character $\omega : \pi \to \{\pm 1\}$. Our motivation herein is the Cappell–Shaneson stable surgery sequence [7, 3.1], whose construction involves certain stable diffeomorphisms. These explicit self-diffeomorphisms lead to a modified version of Wall realization rel ∂X:

$$L_5^s(\mathbf{Z}[\pi^\omega]) \times \mathcal{S}_{\mathrm{DIFF}}^s(X) \longrightarrow \overline{\mathcal{S}}_{\mathrm{DIFF}}^s(X), \tag{1}$$

where $\mathcal{S}$ is the simple smooth structure set and $\overline{\mathcal{S}}$ is the stable structure set. Recall that the equivalence relation on these structure sets is smooth s-bordism of smooth manifold homotopy structures. The actual statement of [7, Theorem 3.1] is sharper in that the amount of stabilization, that is, the number of connected summands of $S^2 \times S^2$, depends only on the rank of a representative of a given element of the odd L-group.

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In the case X is *sufficiently large*, in that it contains a two-sided incompressible smooth 3-submanifold Σ, a periodicity argument using Cappell's decomposition [6, 7] shows that the restriction of the above action on $\mathcal{S}^s_{\mathrm{DIFF}}(X)$ to the summand UNil^s_5 of $L^s_5(\mathbf{Z}[\pi^\omega])$ is free. Therefore for each nonzero element of this exotic UNil-group, there exists a distinct, stable, smooth homotopy structure on X, restricting to a diffeomorphism on ∂X, which is not $\mathbf{Z}[\pi_1(\Sigma)]$-homology splittable along Σ. If Σ is the 3-sphere, the TOP case is [17]. Furthermore, when X is a connected sum of two copies of $\mathbf{RP}^4$, see [15] and [5].

For any $r \geqslant 0$, denote the r**-stabilization of** X by

$$X_r := X\#r(S^2 \times S^2).$$

2. On the topological classification of 4-manifolds

The main result (2.4) of this section is an upper bound on the number of $S^2 \times S^2$ connected summands sufficient for a *stable homeomorphism,* where the fundamental group of X lies in a certain class of good groups. By using Freedman–Quinn surgery [10, §11], if X is also sufficiently large (2.3 for example), each nonzero element ϑ of the UNil-group and simple DIFF homotopy structure $(Y, h : Y \to X)$ pair to form a distinct TOP homotopy structure $(Y_\vartheta, h_\vartheta)$ that represents the DIFF homotopy structure $\vartheta \cdot (Y, h)$ obtained from (1).

2.1. Statement of results

For finite groups π, the theorem's conclusion and the proof's topology are similar to Hambleton–Kreck [12]. However, the algebra is quite different.

Theorem 2.1. *Suppose π is a good group (in the sense of [10]) with orientation character $\omega : \pi \to \{\pm 1\}$. Consider $A := \mathbf{Z}[\pi^\omega]$, a group ring with involution: $\overline{g} = \omega(g)g^{-1}$. Select an involution-invariant subring R of the commutative* Center(A). *Its norm subring is*

$$R_0 \;:=\; \left\{ \sum_i x_i \overline{x}_i \;\middle|\; x_i \in R \right\}.$$

Suppose A is a finitely generated R_0-module, R_0 is noetherian, and the dimension d is finite:

$$d := \dim(\operatorname{maxspec} R_0) < \infty.$$

Now suppose that X is a compact connected TOP *4-manifold with*

$$(\pi_1(X), w_1) = (\pi, \omega)$$

and that it has the form

$$(X, \partial X) = (X_{-1}, \partial X)\#(S^2 \times S^2).$$

If X_r is homeomorphic to Y_r for some $r \geqslant 0$, then X_d is homeomorphic to Y_d.

Here are the class of examples of good fundamental groups promised in the paper's title.

Proposition 2.2. *Suppose π is a finitely generated, virtually abelian group, with any homomorphism ω : $\pi \to \{\pm 1\}$. For some R, the pair (π, ω) satisfies the above hypotheses: π is good, A is a finitely generated*