Topology and its Applications 220 (2017) 65–99

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Diffeological gluing of vector pseudo-bundles and pseudo-metrics on them

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ARTICLE INFO

Article history:
Received 9 May 2016
Received in revised form 25 December 2016
Accepted 1 February 2017
Available online 7 February 2017

MSC:
primary 53C15
secondary 57R35

Keywords:
Diffeology
Diffeological vector pseudo-bundle
Diffeological pseudo-metric

ABSTRACT

Although our main interest here is developing an appropriate analog, for diffeological vector pseudo-bundles, of a Riemannian metric, a significant portion is dedicated to continued study of the gluing operation for pseudo-bundles introduced in [8]. We give more details regarding the behavior of this operation with respect to gluing, also providing some details omitted from [8], and pay more attention to the relations with the spaces of smooth maps. We also show that a usual smooth vector bundle over a manifold that admits a finite atlas can be seen as a result of a diffeological gluing, and thus deduce that its usual dual bundle is the same as its diffeological dual. We then consider the notion of a pseudo-metric, the fact that it does not always exist (which seems to be related to non-local-triviality condition), construction of an induced pseudo-metric on a pseudo-bundle obtained by gluing, and finally, the relation between the spaces of all pseudo-metrics on the factors of a gluing, and on its result. We conclude by commenting on the induced pseudo-metric on the pseudo-bundle dual to the given one.

0. Introduction

For anyone who comes into contact with diffeology [5] (meant initially as an extension of differential geometry, although as of now, other opinions exist) it does not take long to notice that a great number of typical objects do not admit obvious counterparts; there is obviously a diffeological space as the base object, and there are also various types of it endowed with an extra, algebraic, structure, starting with a natural concept of a diffeological vector space and proceeding towards the notion of a diffeological group, diffeological algebra, and so on. But when we try to endow these objects with further structures, even very simple ones, we discover that it is not possible to do so.

The specific instance that we have in mind at the moment is that of a scalar product on a finite-dimensional diffeological vector space. Surprising in its simplicity, it excellently illustrates what has been

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http://dx.doi.org/10.1016/j.topol.2017.02.002
0166-8641/

said in the previous paragraph. Specifically, it has been known for some time (see [5], Ex. 70 on p. 74) that a finite-dimensional diffeological vector space admits a smooth (with respect to its diffeology) scalar product if and only if it is a standard space, which means that, as a vector space, it is isomorphic to $\mathbb{R}^n$ for some n, and its diffeology consists of all usual smooth maps. But for a single vector space this issue is easily resolved: one looks (as was done, for instance, in [7]) for a kind of minimally degenerate smooth symmetric bilinear form on it; there is always one, of rank equal to the dimension of the so-called diffeological dual [12] of the space in question.

The next natural step, then, is to pose the same question for a possible diffeological counterpart of the notion of a Riemannian metric. The corresponding issues become twofold then: for one thing, there is not yet a standard theory of tangent spaces in diffeology (but some constructions do exist, see, for instance, [1], as well as the previous [3], and other references therein), so we are left with looking for a diffeological version of a diffeological metric on an abstract (diffeological version of) vector bundle.

This version appeared originally in [4] (by the author's own admission, the same material is better explained in Chapter 8 of [5]; but the former is the original source). It was later utilized in [11], under the name of a *regular vector bundle*, and then in [1], where it is called a *diffeological vector space over* X, X being the notation for the base space. We call it a *diffeological vector pseudo-bundle*, as was already done in [8]. This concept is an obvious (it is asked that all the operations be smooth in the diffeological sense) extension to the diffeological context of the usual notion of a vector bundle, except that there is no requirement of local triviality. This allows to treat objects that not only carry an unusual smooth structure (that is, a diffeology) but also ones that, from a topological point of view, have a more complicated local structure than that of a Euclidean space.

For objects such as these, there are well-defined notions of the tensor product pseudo-bundle and of the dual pseudo-bundle, see [11]. So a diffeological metric can of course be defined as a smooth section of the tensor product of two duals, symmetric at each point. However, due to what has been said about single vector spaces, the value at a given point cannot, in general, be a scalar product, unless the fibre at this specific point is standard (generally not the case). In this paper we discuss what happens if pointwise we try to take the minimally degenerate value of the prospective section (the approach initiated in [8]).

Acknowledgments Various parts of this work emerged inside of several others — and then moved in here, when it became more sensible for them to do so. I must admit that I maybe would not have kept sensible enough in the meantime if it were not for my colleague and role model Prof. Riccardo Zucchi.

1. Definitions needed

In order to make the paper self-contained, we collect here all the definitions that are used in the rest of the paper.

1.1. Diffeological spaces

The basic notion for diffeology is that of a *diffeological space*, that is, a set endowed with a *diffeology*, a collection of maps called *plots* that play the role of a (version of a) smooth structure.

Diffeological spaces and smooth maps between them A diffeological space is just a set X endowed with a *diffeology*, which is a collection of maps from usual domains to X; three natural conditions must be satisfied.

Definition 1.1. *([9,10])* A **diffeological space** is a pair $(X, \mathcal{D}_X)$ where X is a set and $\mathcal{D}_X$ is a specified collection, also called the **diffeology** of X, of maps $U \to X$ (called **plots**) for each open set U in $\mathbb{R}^n$ and for each $n \in \mathbb{N}$, such that for all open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ the following three conditions are satisfied: