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# Allowing cycles in discrete Morse theory

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A B S T R A C T

Discrete gradient vector fields are combinatorial structures that can be used for accelerating the homology computation of CW complexes, such as simplicial or cubical complexes, by reducing their number of cells. Consequently, they provide a bound for the Betti numbers (the most basic homological information). A discrete gradient vector field can eventually reduce the complex to its minimal form, having as many cells of each dimension as its corresponding Betti number, but this is not guaranteed. Moreover, finding an optimal discrete gradient vector field is an NP-hard problem. We describe here a generalization, which we call Homological Discrete Vector Field (HDVF), which can overcome these limitations by allowing cycles under a certain algebraic condition. In this work we define the HDVF and its associated reduction, we study how to efficiently compute a HDVF, we establish the relation between the HDVF and other concepts in computational homology and we estimate the average complexity of its computation. We also introduce five basic operations for modifying a HDVF, which can also be applied to discrete gradient vector fields.



## 1. Introduction

Morse theory [1] is a tool in differential topology that deduces some information of the topology of a manifold by studying a differentiable function on it. In the late 90s, Robin Forman introduced a discrete version, the discrete Morse theory [2,3], which was defined for CW complexes and discrete functions. Several theorems of Morse theory were translated into the discrete context but, in our opinion, the most notable result was the simplification of a CW complex, which can be used to compute its homology groups.

Homology is an algebraic theory that formalizes the concept of "hole" present in an object. It associates a sequence of abelian groups to an object, whose elements correspond to sums of holes. Up to dimension three, these elements have an easy interpretation. Zero-dimensional holes (elements of the zeroth homology group) correspond to connected components, one-dimensional holes to tunnels or handles and two-dimensional

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holes to cavities. When computing homology, we usually want to find the number of these holes (called the Betti numbers), which are the ranks of the homology groups, and a representative for each hole (called representative cycle of a homology generator).

Homology theory was born more than a century ago and, while all kinds of theoretical results in pure mathematics have been developed, its practical applications have not been exploited until the last 20 years due to its computational complexity. There are applications in dynamical systems [4,5], material science [6,7], electromagnetism [8,9], geometric modeling [10], image understanding [11–14] and sensor networks [15]. The general idea of these applications is that homology is used to analyze and understand high dimensional structures in a rigorous way.

The classical method for computing the homology groups is based on the Smith normal form (SNF) [16], which has super-cubical complexity [17]. Some advances in the computation of the SNF have been achieved, but the best results in computing the homology groups of a complex have been obtained by reducing the number of cells in the complex (see [18–20]).

Among other approaches, let us mention the following two, which are closely related to our work: effective homology theory [21] and discrete Morse theory [3]. Both of them are explained in Section 3.4 and 3.5. The former has the advantage that it "controls" the homology because it contains all the homological information [22]; the latter is very concise and easy to implement. Effective homology theory deals with linear maps which are typically encoded as enormous matrices; discrete Morse theory handles only graphs, but does not always allow us to reduce the complex to its minimal homological expression. The use of reductions (the main concept of effective homology theory) has proved to be successful in the context of image analysis [23,24,12,25] or in a more general setting providing more advanced topological information [26,22].

We aim at finding an intermediate solution, avoiding the respective drawbacks of both of these methods whilst maintaining their advantages. Roughly speaking, discrete Morse theory simplifies a CW complex by establishing arrows on it, hence providing a simpler (in terms of number of cells) complex having the same homological information as the original one. In this article we allow cycles in this "collection of arrows", which is normally forbidden, so that we can go beyond the limits of the classical discrete Morse theory. Moreover, we can control when our approach produces the exact homological information. These allowed cycles must not be confused with the ideas found in [27]. The process of adding these arrows must be simultaneously accompanied by the computation of the linear maps of the effective homology theory, which is unnecessary when there are no cycles. The clearest advantage of our approach over effective homology theory is that we only use linear space for saving these maps, instead of quadratic. Also, our framework works for any dimension, any kind of CW complex and any ring of coefficients.

This article extends the ideas present in [28] under a different formalism, which allows us to find deeper results.

## 2. Previous works

This article somehow creates a new problem instead of solving an existing one. This justifies the shortness of this section.

Discrete Morse theory was introduced in [2,3]. It was then reformulated in terms of matchings in [29,30]. Discrete Morse theory is often used for simplifying a CW complex in order to accelerate the computation of its homology. Thus, it can be seen as an optimization problem, in which one wants to find a discrete gradient vector field (a matching in the Hasse diagram of a CW complex) with as many edges as possible. It was proved that this is an NP-hard problem (see [31,32]). Nevertheless, there has been an extensive research on this optimization problem without aiming at finding a perfect solution in the general case, such as in [31–34]. There has been a parallel and successful research about simplifying a CW complex in [18–20]. These works were recently related to discrete Morse theory in [35], which states that reductions and coreductions are particular strategies for establishing a discrete gradient vector field.