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No interesting sequential groups

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ABSTRACT

We prove that it is consistent with ZFC that no sequential topological groups of intermediate sequential orders exist. This shows that the answer to a 1981 question of P. Nyikos is independent of the standard axioms of set theory. The model constructed also provides consistent answers to several questions of D. Shakhmatov, S. Todorčević and Uzcátegui. In particular, we show that it is consistent with ZFC that every countably compact sequential group is Fréchet–Urysohn.

1. Introduction

A number of areas in mathematics benefit from viewing continuity through the lens of convergence. To investigate the effects of convergence, several classes of spaces have been introduced and studied by set-theoretic topologists. These range from various generalized metric spaces to sequential ones. As a result of these efforts a vast body of classification results and metrization theorems have been developed.

A popular theme has been the study of convergence in the presence of an algebraic structure such as a topological group (see [2] and [17] for a bibliography). One of the first results of this kind, the classical metrization theorem by Birkhoff and Kakutani states that every first countable Hausdorff topological group is metrizable. This establishes a rather unexpected connection between the local and the global properties generally unrelated to each other.

It has been demonstrated by a number of authors, however, that various shades of convergence are, in general, different from each other, even when an algebraic structure is involved (see [1,13,18,20]). A common thread among the majority of these results is the necessity of set-theoretic assumptions beyond ZFC to construct counterexamples.

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A celebrated solution of Malykhin's problem about the metrizability of countable Fréchet groups by Hrušák and Ramos-García [9] is a beautiful validation of the significance of set-theoretic tools in the study of convergence.

A question that is only slightly more recent than Malykhin's problem was asked by P. Nyikos in [13] and deals with the *sequential order* in topological groups. Recall that a space X is *sequential* if whenever $A \subseteq X$ is not closed, there exists a convergent sequence $C \subseteq A$ that converges to a point outside A. This is a rather indirect way of saying that convergent sequences determine the topology of X without supplying any 'constructive means' of describing the closure operator. Such a description is provided by the concept of the *sequential closure* of A: $[A]' =$ "limits of all convergent sequences in A". The next natural step is to recursively define *iterated* sequential closures $[A]_\alpha$, $\alpha \leq \omega_1$ as $[A]_{\alpha+1} = [[A]_\alpha]'$ and $[A]_\alpha = \bigcup\{\, [A]_\beta \mid \beta < \alpha \,\}$ for limit α. It is a quick observation that in sequential spaces $[A]_{\omega_1} = \overline{A}$ and in fact, this property characterizes the class of sequential spaces.

For many spaces it takes only countably many iterations to get the closure of any set. The smallest ordinal $\alpha \leq \omega_1$ such that $[A]_\alpha = \overline{A}$ for every $A \subseteq X$ is called *the sequential order* of X which is written $\alpha = \mathrm{so}(X)$. As a simple illustration of these concepts, sequential spaces are those for which the sequential order is defined and Fréchet (or Fréchet–Urysohn) ones are those whose sequential order is 1.

Simple examples of spaces of *intermediate* (i.e. strictly between 1 and ω_1) sequential orders are plentiful but they all seem to have one common feature: different points of the space have different properties in terms of the sequential closure. This led P. Nyikos to ask the following question.

Question 1 *([13]). Do there exist topological groups of intermediate sequential orders?*

This question and some of its stronger versions were also asked by D. Shakhmatov in [17] (Questions 7.4 (i)–(iii)).

A weak version of this question (for homogeneous and *semi* topological groups, i.e. groups in which the multiplication is continuous in each factor separately) had been answered affirmatively in ZFC (see [6] and [15]).

A consistent positive answer for topological groups was first given in [19] using CH. In [20] it was shown that under CH groups of every sequential order exist.

In this paper we use some of the techniques developed by Hrušák and Ramos-García for their solution of Malykhin's problem to show that extra set-theoretic assumptions are necessary. To be more precise, we construct a model of ZFC in which all sequential groups are either Fréchet or have sequential order ω_1. For countable groups, the result can be viewed as a consistent metrization statement: it is consistent with the axioms of ZFC that all countable sequential groups of sequential order less than ω_1 are metrizable.

As an aside, we show how the same model provides a consistent answer to a question of D. Shakhmatov about the structure of countably compact sequential groups.

2. Preliminaries

We use standard set-theoretic terminology, see [11]. By a slight abuse of notation we sometimes treat sequences as sets of points in their range. Basic facts about topological groups can be found in [2]. All spaces are assumed to be regular.

Following [9] define *Laver–Mathias–Prikry forcing* $\mathbb{L}_\mathcal{F}$ associated to a free filter $\mathcal{F}$ on ω as the set of those trees $T \subseteq \omega^{<\omega}$ for which there is an $s_T \in T$ (the *stem of* T) such that for all $s \in T$, $s \subseteq s_T$ or $s_T \subseteq s$ and such that for all $s \in T$ with $s \supseteq s_T$ the set $\mathrm{succ}_T(s) = \{\, n \in \omega \mid s^\frown n \in T \,\} \in \mathcal{F}$ ordered by inclusion.

Full details of proofs of various properties of $\mathbb{L}_\mathcal{F}$ can be found in [9], here we only present the statements directly used in the arguments in this paper.