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Filter-Laver measurability

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ABSTRACT

We study σ-ideals and regularity properties related to the "filter-Laver" and "dual-filter-Laver" forcing partial orders. An important innovation which enables this study is a dichotomy theorem proved recently by Miller [19].

1. Introduction

In this paper, F will always be a filter on ω (or a suitable countable set), containing at least the cofinite sets. We will use F^- to refer to the ideal of all $a \subseteq \omega$ such that $\omega \setminus a \in F$, and F^+ to the collection of $a \subseteq \omega$ such that $a \notin F^-$. Cof and Fin denote the filter of cofinite subsets of ω and the ideal of finite subsets of ω, respectively.

Definition 1.1. An F*-Laver tree* is a tree $T \subseteq \omega^{<\omega}$ such that for all $\sigma \in T$ extending stem(T), $\mathrm{Succ}_T(\sigma) \in F$. An F^+*-Laver-tree* is a tree $T \subseteq \omega^{<\omega}$ such that for all $\sigma \in T$ extending stem(T), $\mathrm{Succ}_T(\sigma) \in F^+$. We use $\mathbb{L}_F$ and $\mathbb{L}_{F^+}$ to denote the partial orders of F-Laver and F^+-Laver trees, respectively, ordered by inclusion.

If $F = \mathsf{Cof}$ then $\mathbb{L}_{F^+}$ is the standard *Laver forcing* $\mathbb{L}$, and $\mathbb{L}_F$ is (a version of) the standard *Hechler forcing* $\mathbb{D}$. Both $\mathbb{L}_F$ and $\mathbb{L}_{F^+}$ have been used as forcing notions in the literature, see, e.g., [9]. As usual, the generic real added by these forcings can be defined as the limit of the stems of conditions in the generic filter. It is easy to see that in both cases, this generic real is dominating. It is also known that if F is not an ultrafilter, then $\mathbb{L}_F$ adds a Cohen real, and if F is an ultrafilter, then $\mathbb{L}_F$ adds a Cohen real if and only if F is not a *nowhere dense ultrafilter* (see Definition 4.9). Moreover, $\mathbb{L}_F$ is σ-centered and hence satisfies the ccc, and it is known that $\mathbb{L}_{F^+}$ satisfies Axiom A (see [9, Theorem] and Lemma 2.5 (3)).

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In this paper, we consider σ-ideals and regularity properties naturally related to $\mathbb{L}_F$ and $\mathbb{L}_{F^+}$, and study the regularity properties for sets in the low projective hierarchy, following ideas from [3,12,14]. An important technical innovation is a dichotomy theorem proved recently by Miller in [19] (see Theorem 3.6), which allows us to simplify the σ-ideal for $\mathbb{L}_{F^+}$ when restricted to Borel sets, while having a $\mathbf{\Sigma}^1_2$ definition regarding the membership of Borel sets in it.

One question may occur to the reader of this paper: why are we not considering the *filter-Mathias* forcing alongside the filter-Laver forcing, when clearly the two forcing notions (and their derived σ-ideals and regularity properties) are closely related? The answer is that, although the basic results from Section 2 do indeed hold for filter-Mathias, there is no corresponding dichotomy theorem like Theorem 3.6. In fact, by a result of Sabok [21], even the σ-ideal corresponding to the *standard* Mathias forcing is not a $\mathbf{\Sigma}^1_2$-ideal on Borel sets, implying that even in this simple case, there is no hope of a similar dichotomy theorem. It seems that in the Mathias case, a more subtle analysis is required.

In Section 2 we give the basic definitions and prove some easy properties. In Section 3 we present Miller's dichotomy and the corresponding σ-ideal. In Section 4 we study direct relationships that hold between the regularity properties regardless of the complexity of F, whereas in Section 5 we prove stronger results under the assumption that F is an analytic filter.

2. ($\mathbb{L}_F$)- and ($\mathbb{L}_{F^+}$)-measurable sets

In [12], Ikegami provided a natural framework for studying σ-ideals and regularity properties related to tree-like forcing notions, generalising the concepts of *meager* and *Baire property*. This concept proved to be very useful in a number of circumstances, see, e.g., [14,16,15].

Definition 2.1. Let $\mathbb{P}$ be $\mathbb{L}_F$ or $\mathbb{L}_{F^+}$ and let $A \subseteq \omega^\omega$.

1. $A \in \mathcal{N}_\mathbb{P}$ iff $\forall T \in \mathbb{P}\, \exists S \leq T\, ([S] \cap A = \varnothing)\}$.
2. $A \in \mathcal{I}_\mathbb{P}$ iff A is contained in a countable union of sets in $\mathcal{N}_\mathbb{P}$.
3. A is $\mathbb{P}$-*measurable* iff $\forall T \in \mathbb{P}\, \exists S \leq T\, ([S] \subseteq^* A$ or $[S] \cap A =^* \varnothing)$, where $\subseteq^*$ and $=^*$ stands for "modulo a set in $\mathcal{I}_\mathbb{P}$".

Lemma 2.2. *The collection $\{[T] \mid T \in \mathbb{L}_F\}$ forms a topology base. The resulting topology refines the standard topology and the space satisfies the Baire category theorem (i.e., $[T] \notin \mathcal{I}_{\mathbb{L}_F}$ for all $T \in \mathbb{L}_F$).*

Proof. Clearly, for all $S, T \in \mathbb{L}_F$ the intersection $S \cap T$ is either empty or an $\mathbb{L}_F$-condition. A basic open set in the standard topology trivially corresponds to a tree in $\mathbb{L}_F$. For the Baire category theorem, let A_n be nowhere dense and, given an arbitrary $T \in \mathbb{L}_F$, build a sequence $T = T_0 \geq T_1 \geq T_2 \geq \dots$ with strictly increasing stems such that $[T_n] \cap A_n = \varnothing$ for all n. Then the limit of the stems is an element in $[T] \setminus \bigcup_n A_n$. $\square$

We use $\tau_{\mathbb{L}_F}$ to denote the topology on ω^ω generated by $\{[T] \mid T \in \mathbb{L}_F\}$. Clearly $\mathcal{N}_{\mathbb{L}_F}$ is the collection of $\tau_{\mathbb{L}_F}$-nowhere dense sets and $\mathcal{I}_{\mathbb{L}_F}$ the collection of $\tau_{\mathbb{L}_F}$-meager sets. Moreover, we recall the following fact, which is true in arbitrary topological spaces (the proof is similar to [13, Theorem 8.29]):

Fact 2.3. *Let $\mathcal{X}$ be any topological space, and $A \subseteq \mathcal{X}$. Then the following are equivalent:*

1. *A satisfies the Baire property.*
2. *For every basic open O there is a basic open $U \subseteq O$ such that $U \subseteq^* A$ or $U \cap A =^* \varnothing$, where $\subseteq^*$ and $=^*$ refer to "modulo meager".*

In particular, $A \subseteq \omega^\omega$ is $\mathbb{L}_F$-measurable iff A satisfies the $\tau_{\mathbb{L}_F}$-Baire property.