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The paucity of universal compacta in cohomological dimension

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ABSTRACT

Let $\mathcal{C}$ be a class of spaces. An element $Z \in \mathcal{C}$ is called *universal* for $\mathcal{C}$ if each element of $\mathcal{C}$ embeds in Z. It is well-known that for each $n \in \mathbb{N}$, there exists a universal element for the class of metrizable compacta X of (covering) dimension $\dim X \leq n$. The situation in cohomological dimension over an abelian group G, denoted $\dim_G$, is almost the opposite. Our results will imply in contradistinction that for each nontrivial abelian group G and for $n \geq 2$, there exists no universal element for the class of metrizable compacta X with $\dim_G X \leq n$.

1. Introduction

For any class $\mathcal{C}$ of spaces, an element $Z \in \mathcal{C}$ is called *universal* for $\mathcal{C}$ if each element of $\mathcal{C}$ embeds in Z. It is a standard fact in dimension theory (1.11.7 of [8]) that for each $n \geq -1$ there exists a universal object for the class of metrizable compacta X with $\dim X \leq n$. We are going to prove that for $\dim_G$, the theory of cohomological dimension over a nontrivial abelian group G [13], such universal objects do not exist for $n \geq 2$.

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There is a very useful approach to the theories of covering dimension dim and cohomological dimension $\dim_G$ over an abelian group G. If K is a CW-complex and X is a space, then one says that K is an *absolute extensor* for X, $K \in \mathrm{AE}(X)$, or that X is an *absolute co-extensor* for K, $X\tau K$, if for each closed subset A of X and map $f : A \to K$, there exists a map $F : X \to K$ such that F is an extension of f. This is the fundamental notion of extension theory. For metrizable or compact Hausdorff spaces X one can say that $\dim X \leq n$ if and only if $X\tau S^n$ while $\dim_G X \leq n$ if and only if $X\tau K(G, n)$.[1]

Let K be a CW-complex and $\mathcal{C}$ the class of metrizable compacta X with $X\tau K$. If it is true that for every subset $\mathcal{B}$ of $\mathcal{C}$, the topological sum $\bigsqcup \mathcal{B}$ has the property that the Stone–Čech compactification $\beta(\bigsqcup \mathcal{B})\tau K$, then it can be shown that $\mathcal{C}$ has a universal element. A model for the proof of such a statement can be found in Problem 7.4.16 of [7]. It uses the Mardešić Factorization Theorem for S^n, i.e., for covering dimension. Such a factorization theorem applies, however, for all CW-complexes K by the main result of [15]. Let us point out that if K is finite or even finitely dominated, then by Lemma 1.10 of [15], $\beta(\bigsqcup \mathcal{B})\tau K$ is true. So universal metrizable compacta X of the type $X\tau K$ exist for such K. In [15], using this lemma, we gave a proof of this fact using the more general factorization theorem (see Corollary 1.9 of [15]). We will return to these ideas in Section 19. Let us focus now on the main theme of this paper.

In the middle 1980's the author began to investigate the question of whether universal compacta exist for the class $\mathcal{C}$ of metrizable compacta X with $\dim_{\mathbb{Z}} X \leq n$ and $n \geq 2$. In 1988 A. Dranishnikov [1] proved that the property $\beta(\bigsqcup \mathcal{B})\tau K$ for every subset $\mathcal{B} \subset \mathcal{C}$ does not hold true in case $K = K(\mathbb{Z}, 4)$. Hence although for every subset $\mathcal{B} \subset \mathcal{C}$, $\bigsqcup \mathcal{B}$ is a metrizable space with $\dim_{\mathbb{Z}}(\bigsqcup \mathcal{B}) \leq 4$, there exists a sub-collection $\mathcal{B} \subset \mathcal{C}$ with $\dim_{\mathbb{Z}} \beta(\bigsqcup \mathcal{B}) > 4$. Based on this and some other rather "thin" evidence, J. Dydak and J. Mogilski [6] (see the Remark on page 947) conjectured that universal compacta for the class of metrizable compacta X with $\dim_{\mathbb{Z}} X \leq 4$ do not exist. We raised this issue in a more general sense as Question 8.4 in [17].

Our Main Result is Theorem 2.2, and its Main Application is Corollary 18.1. The latter states that for every nontrivial abelian group G and $n \geq 2$, there is no universal element in the class of metrizable compacta X with $\dim_G X \leq n$. This of course proves the recently mentioned conjecture of Dydak and Mogilski. The technique of proof is novel in that we employ a direct system $\mathbf{X}$ over an uncountable directed indexing set, consisting of metrizable compacta and not necessarily injective connecting maps where $\mathbf{X}$ has certain important properties. Such systems (with injective connecting maps) and their direct limits X have been studied with respect to extension theory in [16,11,12]. We, however, will need a different result (Theorem 15.1) in order to detect the essential extension-theoretic properties, and in particular the normality, of such an X. In addition, it will be imperative that the direct limit space X be pseudo-compact, i.e., every map of X to $\mathbb{R}$ has bounded image; we shall see that pseudo-compact spaces are "compact enough" for our needs.

We are going to rely on and improve Theorem 7.6 of [17] which is independent of the research in this paper. Given a CW-complex K, it involves a relationship between the K-absolute co-extensor property of the Stone–Čech compactification of the topological sum of a collection of compact metrizable spaces and K-invertibility (a type of map lifting property). When $K = K(G, n)$ for $n \geq 2$, this will prove that the suspected link between Stone–Čech compactification and the Dydak–Mogilski conjecture of [6] about the nonexistence of universal compacta is genuine. The large class of examples for which our Main Result applies will be extracted from the work of M. Levin [14]. The strategy behind this is revealed in Section 2.

In the final section of this paper, we will provide some examples in $\dim_G \leq 1$ for which universal compacta exist and will state a conjecture about universal compacta of arbitrary weight.

2. Strategy

We are going to prove Theorem 2.2 which alleges the equivalence of nine statements. It adds one new equivalence (i) to the list of eight, (a)–(h), that appear in Theorem 7.6 of [17]. To establish the equivalence

[1] By $K(G, n)$ we mean any Eilenberg–MacLane CW-complex of type (G, n).