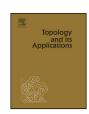


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Completeness of locally k_{ω} -groups and related infinite-dimensional Lie groups



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ABSTRACT

Recall that a topological space X is said to be a k_{ω} -space if it is the direct limit of an ascending sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact Hausdorff topological spaces. If each point in a Hausdorff space X has an open neighbourhood which is a k_{ω} -space, then X is called locally k_{ω} . We show that a topological group is complete whenever the underlying topological space is locally k_{ω} . As a consequence, every infinite-dimensional Lie group modelled on a Silva space is complete.

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1. Introduction and statement of the results

Recall that a k_{ω} -space is a Hausdorff topological space X which carries the direct limit topology for an ascending sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets $K_n \subseteq X$ with $\bigcup_{n \in \mathbb{N}} K_n = X$ (see [8,13]; cf. [22] and, with different terminology, [24]). A topological group is called a k_{ω} -group if its underlying topological space is a k_{ω} -space. Hunt and Morris [17] showed that every k_{ω} -group is Weil complete, viz., complete in its left uniformity (cf. [29] for the case of abelian k_{ω} -groups; see also [3] for a recent proof). The current work is devoted to generalizations and applications of this fact, with a view towards examples in infinite-dimensional Lie theory.

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Following [13], a Hausdorff space X is called *locally* k_{ω} if each $x \in X$ has an open neighbourhood $U \subseteq X$ which is a k_{ω} -space in the induced topology. A topological group G is called locally k_{ω} if its underlying topological space is locally k_{ω} . Since every locally k_{ω} group has an open subgroup which is a k_{ω} -group [13, Proposition 5.3], the Hunt-Morris Theorem implies the following:

Proposition 1.1. Every locally k_{ω} topological group is Weil complete. \square

We consider Lie groups modelled on arbitrary real or complex Hausdorff locally convex topological vector spaces (as in [26] and [14]),¹ based on the differential calculus in locally convex spaces known as Keller's C_c^{∞} -theory [18]. Since the smooth maps under consideration are, in particular, continuous, the Lie groups we consider have continuous group operations. They can therefore be regarded as Hausdorff topological groups, and we can ask when they are complete. The following observation (proved in Section 2) is essential.

Proposition 1.2. If a Lie group G is modelled on a locally convex space E which is a k_{ω} -space, then G is locally k_{ω} , and its identity component G_e is k_{ω} .

Propositions 1.1 and 1.2 entail a simple completeness criterion:

Corollary 1.3. Every Lie group modelled on a k_{ω} -space is Weil complete. \square

Recall that a locally convex space E is called a *Silva space* if it is the locally convex direct limit $E = \lim_{n \to \infty} E_n$ of an ascending sequence $E_1 \subseteq E_2 \subseteq \cdots$ of Banach spaces, such that each inclusion map $E_n \to E_{n+1}$ is a compact operator (cf. [7]). It is well known that every Silva space is a k_{ω} -space (see, e.g., [11, Example 9.4]). Thus Corollary 1.3 entails:

Corollary 1.4. Lie groups modelled on Silva spaces are Weil complete. \Box

Until recently, little was known on the completeness properties of infinite-dimensional Lie groups, except for the classical fact that Lie groups modelled on Banach spaces are Weil complete (see Proposition 1 in [2, Chapter III, §1.1]). In 2016, Weil completeness was established for many classes of infinite-dimensional Lie groups [12], but some examples modelled on Silva spaces could not be treated. The current paper closes this gap, as Corollary 1.4 establishes Weil completeness for the latter. Section 3 compiles a list of infinite-dimensional Lie groups which are modelled on Silva spaces and hence Weil complete (by Corollary 1.4). In particular, we find:

- For each compact real analytic manifold M, the Lie group $\mathrm{Diff}^{\omega}(M)$ of all real analytic diffeomorphisms $\phi \colon M \to M$ is Weil complete;
- For each finite-dimensional Lie group G and M as before, the Lie group $C^{\omega}(M,G)$ of all real analytic maps $f: M \to G$ is Weil complete.

Being modelled on Silva spaces, the examples we consider are locally k_{ω} (by Proposition 1.2), which is sufficient to conclude Weil completeness. Yet, it is natural to ask whether the groups in question are not only locally k_{ω} , but, actually, k_{ω} -groups. Results in Section 4 subsume:

Proposition 1.5. For each compact real analytic manifold M, the Lie group $\mathrm{Diff}^{\omega}(M)$ is a k_{ω} -group. Moreover, $C^{\omega}(M,G)$ is a k_{ω} -group for each σ -compact finite-dimensional Lie group G, and M as before.

¹ Compare also [23] (for Lie groups modelled on sequentially complete spaces).

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