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Completeness of locally k_ω-groups and related infinite-dimensional Lie groups

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A B S T R A C T

Recall that a topological space X is said to be a k_ω*-space* if it is the direct limit of an ascending sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact Hausdorff topological spaces. If each point in a Hausdorff space X has an open neighbourhood which is a k_ω-space, then X is called *locally* k_ω. We show that a topological group is complete whenever the underlying topological space is locally k_ω. As a consequence, every infinite-dimensional Lie group modelled on a Silva space is complete.

1. Introduction and statement of the results

Recall that a k_ω*-space* is a Hausdorff topological space X which carries the direct limit topology for an ascending sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets $K_n \subseteq X$ with $\bigcup_{n\in\mathbb{N}} K_n = X$ (see [8,13]; cf. [22] and, with different terminology, [24]). A topological group is called a k_ω*-group* if its underlying topological space is a k_ω-space. Hunt and Morris [17] showed that every k_ω-group is Weil complete, viz., complete in its left uniformity (cf. [29] for the case of abelian k_ω-groups; see also [3] for a recent proof). The current work is devoted to generalizations and applications of this fact, with a view towards examples in infinite-dimensional Lie theory.

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Following [13], a Hausdorff space X is called *locally* k_ω if each $x \in X$ has an open neighbourhood $U \subseteq X$ which is a k_ω-space in the induced topology. A topological group G is called locally k_ω if its underlying topological space is locally k_ω. Since every locally k_ω group has an open subgroup which is a k_ω-group [13, Proposition 5.3], the Hunt–Morris Theorem implies the following:

Proposition 1.1. *Every locally k_ω topological group is Weil complete.* □

We consider Lie groups modelled on arbitrary real or complex Hausdorff locally convex topological vector spaces (as in [26] and [14]),[1] based on the differential calculus in locally convex spaces known as Keller's C^∞_c-theory [18]. Since the smooth maps under consideration are, in particular, continuous, the Lie groups we consider have continuous group operations. They can therefore be regarded as Hausdorff topological groups, and we can ask when they are complete. The following observation (proved in Section 2) is essential.

Proposition 1.2. *If a Lie group G is modelled on a locally convex space E which is a k_ω-space, then G is locally k_ω, and its identity component G_e is k_ω.*

Propositions 1.1 and 1.2 entail a simple completeness criterion:

Corollary 1.3. *Every Lie group modelled on a k_ω-space is Weil complete.* □

Recall that a locally convex space E is called a *Silva space* if it is the locally convex direct limit $E = \varinjlim E_n$ of an ascending sequence $E_1 \subseteq E_2 \subseteq \cdots$ of Banach spaces, such that each inclusion map $E_n \to E_{n+1}$ is a compact operator (cf. [7]). It is well known that every Silva space is a k_ω-space (see, e.g., [11, Example 9.4]). Thus Corollary 1.3 entails:

Corollary 1.4. *Lie groups modelled on Silva spaces are Weil complete.* □

Until recently, little was known on the completeness properties of infinite-dimensional Lie groups, except for the classical fact that Lie groups modelled on Banach spaces are Weil complete (see Proposition 1 in [2, Chapter III, §1.1]). In 2016, Weil completeness was established for many classes of infinite-dimensional Lie groups [12], but some examples modelled on Silva spaces could not be treated. The current paper closes this gap, as Corollary 1.4 establishes Weil completeness for the latter. Section 3 compiles a list of infinite-dimensional Lie groups which are modelled on Silva spaces and hence Weil complete (by Corollary 1.4). In particular, we find:

- For each compact real analytic manifold M, the Lie group $\mathrm{Diff}^\omega(M)$ of all real analytic diffeomorphisms $\phi: M \to M$ is Weil complete;
- For each finite-dimensional Lie group G and M as before, the Lie group $C^\omega(M, G)$ of all real analytic maps $f: M \to G$ is Weil complete.

Being modelled on Silva spaces, the examples we consider are locally k_ω (by Proposition 1.2), which is sufficient to conclude Weil completeness. Yet, it is natural to ask whether the groups in question are not only locally k_ω, but, actually, k_ω-groups. Results in Section 4 subsume:

Proposition 1.5. *For each compact real analytic manifold M, the Lie group $\mathrm{Diff}^\omega(M)$ is a k_ω-group. Moreover, $C^\omega(M, G)$ is a k_ω-group for each σ-compact finite-dimensional Lie group G, and M as before.*

[1] Compare also [23] (for Lie groups modelled on sequentially complete spaces).