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Strong domination by countable and second countable spaces

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We show that, for a Lindelöf Σ-space X, if $C_p(X,[0,1])$ is strongly dominated by a second countable space, then X is countable. Under Martin's Axiom we prove that there exists a countable space Z that strongly dominates the complement of the diagonal of any first countable compact space. In particular, strong domination by a countable space of the complement of the diagonal of a compact space X need not imply metrizability of X. It turns out that the same countable space Z strongly dominates $C_p(X)$ for an uncountable space X. Our results solve several published open problems.

1. Introduction

If X and Y are topological spaces and $f : Y \to X$ is a continuous onto map, then letting $F_K = f(K)$ for every compact set $K \subset Y$, we obtain a compact cover $\mathcal{F} = \{F_K : K \text{ is a compact subset of } Y\}$ of the space X ordered by the family $\mathcal{K}(Y)$ of all compact subsets of Y in the sense that $K \subset L$ implies $F_K \subset F_L$. Now, if X is not necessarily a continuous image of Y but there exists a compact cover of X ordered by $\mathcal{K}(Y)$, then it is said that *the space X is dominated by Y*. We already saw that domination by a space Y is a weaker concept than being a continuous image of Y.

A space X is strongly dominated by a space Y if there exists a compact cover $\mathcal{F}$ of X ordered by $\mathcal{K}(Y)$ that swallows all compact subsets of X, i.e., for any compact set $E \subset X$, there exists $K \in \mathcal{K}(Y)$ such that $E \subset F_K$. It is easy to see that strong domination by Y is a generalization of being a perfect continuous image

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of Y. The importance of the notion of domination can be seen from the fact that every K-analytic space is dominated by the space $\mathbb{P}$ of the irrationals and Lindelöf Σ-spaces are dominated by second countable ones.

The first important bulk of results on domination and strong domination comes from Functional Analysis. Valdivia studied linear topological spaces dominated by $\mathbb{P}$ and Talagrand proved in [12] that $C_p(X)$ must be K-analytic if X is compact and $C_p(X)$ is dominated by $\mathbb{P}$. Tkachuk showed in [13] that compactness of X can be omitted in Talagrand's result. In [3] Cascales and Orihuela obtained some metrization theorems for compact subsets of function spaces in terms of strong domination by $\mathbb{P}$. Christensen's work in Descriptive Set Theory [1] had, as a purely topological byproduct, a theorem that states that a second countable space is strongly dominated by $\mathbb{P}$ if and only if it is Polish.

Tkachuk established in [13] that $C_p(X)$ is strongly dominated by $\mathbb{P}$ if and only if X is countable and discrete. Cascales, Orihuela and Tkachuk showed that $C_p(X)$ is dominated by a second countable space if and only if it has the Lindelöf Σ-property and asked whether strong domination of $C_p(X)$ by a second countable space implies countability of X. This question was answered positively by Guerrero Sánchez and Tkachuk in [7] who asked in the same paper what happens if $C_p(X, [0, 1])$ is strongly dominated by a second countable space. In this case the space X need not be countable because $C_p(X, [0, 1]) = [0, 1]^X$ is even compact for any discrete space X. So far there is no conjecture on how to characterize strong domination of $C_p(X, [0, 1])$ by a second countable space but it was asked in [7] whether X must be countable if it has the Lindelöf Σ-property.

In this paper we prove that a Lindelöf Σ-space X is, indeed, countable if $C_p(X, [0, 1])$ is strongly dominated by a second countable space; this solves Question 4.9 from the paper [7]. The fact that a compact space X is metrizable if $(X \times X)\backslash\Delta_X$ is strongly dominated by a second countable space (see Theorem 3.11 of [3]) was the motivation for Guerrero Sánchez and Tkachuk to ask in [6] whether the same conclusion is true if $(X \times X)\backslash\Delta_X$ is strongly dominated by a space with a countable network. In this paper we establish that, under Martin's Axiom, even strong domination of $(X \times X)\backslash\Delta_X$ by a countable space does not imply metrizability of a compact space X. This gives a consistent solution of Questions 4.4–4.13 from the paper [6].

2. Notation and terminology

All spaces are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\backslash\{\emptyset\}$. Let $\tau(A, X) = \{U \in \tau(X) : A \subset U\}$ for any $A \subset X$; if $x \in X$, then we write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. As usual, $\mathbb{R}$ is the set of reals; the set $\omega\backslash\{0\}$ is denoted by $\mathbb{N}$ and $\mathbb{I} = [0, 1] \subset \mathbb{R}$; we will also need the two-point discrete space $\mathbb{D} = \{0, 1\} \subset \mathbb{R}$.

If X is a space, then $\Delta_X = \{(x, x) : x \in X\}$ is its diagonal and $\mathcal{K}(X)$ is the family of all compact subsets of X. Say that a family $\mathcal{F}$ of subsets of a space X is *a network with respect to a cover* $\mathcal{C}$ if for any $C \in \mathcal{C}$ and $U \in \tau(C, X)$ there exists $F \in \mathcal{F}$ such that $C \subset F \subset U$. A space X is *Lindelöf* Σ (or has *the Lindelöf* Σ*-property*) if there exists a countable family $\mathcal{F}$ of subsets of X such that $\mathcal{F}$ is a network with respect to a compact cover $\mathcal{C}$ of the space X. If there exists a countable family $\mathcal{F}$ of subsets of X such that $\mathcal{F}$ is a network with respect to $\mathcal{K}(X)$, then X is called *an* $\aleph_0$*-space*. The space X has *the Banakh property*, if there is a countable family $\mathcal{F}$ of nowhere dense subsets of X such that for any $K \in \mathcal{K}(X)$ there exists $F \in \mathcal{F}$ with $K \subset F$.

A family $\mathcal{N}$ is a network in a space X if for every $U \in \tau(X)$, there exists a family $\mathcal{N}' \subset \mathcal{N}$ such that $U = \bigcup \mathcal{N}'$. The spaces with a countable network are also called *cosmic*. A continuous map $f : X \to Y$ is $\mathbb{R}$*-quotient* if for any map $g : Y \to \mathbb{I}$, it follows from continuity of $g \circ f$ that g is continuous. A space X is called *stable* if for any continuous image Y of the space X and any continuous injection of Y into a space Z, we have $nw(Y) \le w(Z)$.

Given a set X, let $\exp(X) = \{Y : Y \subset X\}$. If X is a space and $f : X \to Y$ is a continuous map, then a family $\mathcal{N} \subset \exp(X)$ is *a network of* f if for any $x \in X$ and $U \in \tau(f(x), Y)$ there exists $N \in \mathcal{N}$ such