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Consistent Smyth powerdomains of topological spaces and quasicontinuous domains ☆

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ABSTRACT

The consistent Smyth powerdomain $RQ_C(X)$ of a topological space X means the family of all nonempty relatively compact-connected saturated subsets of X, ordered by the reverse inclusion and endowed with the upper Vietoris topology. In this paper, we study properties of consistent Smyth powerdomains of certain topological spaces (especially, quasicontinuous domains equipped with the Scott topology) from topological, order theoretical and categorical aspects. Main results are: (i) a topological space X is sober iff $RQ_C(X)$ is sober; (ii) if X is locally compact-connected, well-filtered and coherent, then $RQ_C(X)$ is coherent; (iii) every dcpo equipped with the Scott topology is locally connected, and if a dcpo L is finitely up-generated, locally compact, well-filtered and coherent, then $RQ_C(L)$ is a Lawson compact L-domain; (iv) if L is a quasicontinuous domain (resp., a Lawson compact quasicontinuous domain, a quasialgebraic domain), then $RQ_C(L)$ is a continuous dcpo-$\wedge^{\uparrow}$-semilattice (resp., a Lawson compact L-domain, an algebraic dcpo-$\wedge^{\uparrow}$-semilattice); (v) it is proved that $RQ_C(L)$ is a free continuous dcpo-$\wedge^{\uparrow}$-semilattice over a quasicontinuous domain L.

1. Introduction

Powerdomain constructions provide mathematical models for semantics of nondeterministic programming languages in theoretical computer science. There are many kinds of powerdomain constructions in the literature, such as the Plotkin powerdomain (or, the convex powerdomain) proposed by Plotkin [16], the Smyth powerdomain (or, the upper powerdomain) [18], the Hoare powerdomain (or, the lower powerdomain) proposed by Smyth [19], and the probabilistic powerdomain [10]. The Smyth powerdomain and the Plotkin powerdomain were initially proposed for domains, while the Hoare powerdomain initially for dcpos. So far, a large volume of work on these powerdomain constructions has been done [1,2,6,7,11,17]. It is worthy of

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noting that classical Smyth powerdomain constructions have been generalized to the topological setting, especially, to the setting of quasicontinuous domains [3] equipped with the Scott topology. Kou and Luo in [11] investigated adjunctions between the category of quasicontinuous domains and its subcategories via the Smyth powerdomain. Heckmann and Keimel in [8] obtained several properties of the Smyth powerdomain $Q(X)$ of a topological space X and characterized quasicontinuous domains by their Smyth powerdomain constructions. In terms of the Smyth powerdomain construction for topological spaces, Goubault-Larrecq and Jung [5], Lawson and Xi [12] independently proved that the structures QRB-domains [4], QFS-domains [13] and the Lawson compact quasicontinuous domains are the same things.

Recently, Yuan and Kou in [21] introduced a new powerdomain construction called the consistent Smyth powerdomain. For a topological space X, the consistent Smyth powerdomain $RQ_C(X)$ of X means the family of all nonempty relatively compact-connected and saturated subsets of X, ordered by the reverse inclusion and endowed with the upper Vietoris topology. They showed that the consistent Smyth powerdomain over a domain is a continuous dcpo-$\wedge^\uparrow$-semilattice and that the new construction is a free algebra construction over domains with consistent meet operators $\wedge^\uparrow$, revealing that the new construction shares important merits of the classical Smyth powerdomain construction.

Now some questions naturally arise: whether the consistent Smyth powerdomain of quasicontinuous domains are domains or not? To what extent the consistent Smyth powerdomain of a topological space (resp., a dcpo equipped with the Scott topology) reflects (or, is determined by) properties of the original space (resp., dcpo)? Is $RQ_C(L)$ still a free continuous dcpo-$\wedge^\uparrow$-semilattice over a quasicontinuous domain L?

In this paper, we will give some solutions to these questions. We will show that the consistent Smyth powerdomain over a quasicontinuous domain is a continuous dcpo-$\wedge^\uparrow$-semilattice, especially a domain. We will prove that if a topological space X is finitely up-generated, locally compact-connected, well-filtered and coherent, then the consistent Smyth powerdomain $RQ_C(X)$ is a Lawson compact L-domain. It is also obtained that the consistent Smyth powerdomain $RQ_C(X)$ is sober iff X is sober. Finally, it is proved that $RQ_C(L)$ is indeed a free continuous dcpo-$\wedge^\uparrow$-semilattice over a quasicontinuous domain L.

2. Preliminaries

We recall some basic concepts and results which will be used in the sequel. Most of them come from [2] and [21].

Let L be a poset. For $D \subseteq L$, we use $\vee D$ (resp., $\wedge D$) to denote the supremum (resp., infimum) of D if it exists. A poset is called a *directed complete poset* (dcpo, in short) if every directed subset in it has a supremum. A subset A of L is called *consistent* if A has an upper bound in L, i.e., $A \subseteq \downarrow x$ for some $x \in L$. A poset is called *bounded complete* if every subset that is bounded above has a least upper bound. It is easy to see that L is bounded complete iff every consistent subset of L has a supremum. In particular, a bounded complete poset has a bottom, the least upper bound of the empty set. We say that x *approximates* y in L, or x is *way-below* y, written as $x \ll y$ if for any directed set $D \subseteq L$ with existing $\vee D \geq y$, there is some $d \in D$ s.t. $x \leq d$. An element $x \in L$ is called *compact* if $x \ll x$. A poset L is said to be *continuous* (resp., *algebraic*) if every element is the directed supremum of (resp., compact) elements that approximate it. A continuous dcpo is often called a *domain*. A domain L is called an *L-domain* if every its principal ideal $\downarrow x$ is a complete lattice (in its induced order) for all $x \in L$, or equivalently, every consistent subset of L has an infimum. A domain is called a *bounded complete domain* if it is bounded complete.

A subset $U \subseteq L$ is *Scott open* iff (i) $U = \uparrow U$, and (ii) for any directed subset $D \subseteq L$, $\vee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\vee D$ exists. The collection of all Scott open sets of L forms a topology, called the *Scott topology* of L and denoted by $\sigma(L)$. The topology on L generated by $\{L \backslash \uparrow x : x \in L\}$ as a subbase is called the *lower topology*, which is denoted by $\omega(L)$. And the *Lawson topology* $\lambda(L)$ is the coarsest topology containing $\sigma(L) \cup \omega(L)$. For a topological space X, a subset $A \subseteq X$ is said to be *irreducible* if for any finite family $\{C_i\}_{i \in F}$ of closed sets, whenever $A \subseteq \cup_{i \in F} C_i$, then $A \subseteq C_i$ for some $i \in F$. A topological space