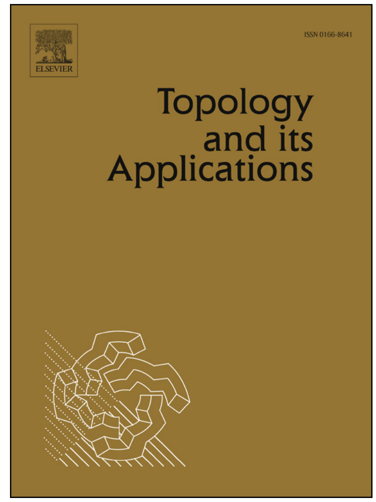


# Tropical convexity in Riesz spaces

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# TROPICAL CONVEXITY IN RIESZ SPACES

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Abstract. Various properties of tropical convex sets of the positive cone of a Riesz space are established. Under suitable, but standard assumptions, we show that they have fixed points properties, for single valued and multivalued maps, analoguous to those associated to the usual convex sets in topological vector spaces. Selection theorems and extension theorems similar to those of Michael and Dugundji also hold and, in the appropriate setting, tropical convex sets are absolute extensors for the class of metric spaces and absolute retracts. We also give some applications to standard problems of mathematical economics: Nash equilibria, existence of elements for non transitive preference relations and existence of equilibria for abstract economies in Riesz spaces under tropical convexity assumptions. This work can be conceived as a contribution to infinite dimensional idempotent analysis.

**MSC:** 14T99, 46A40, 54H25, 54C55, 54C65, 62P20.

## 1. Introduction

At the core of "Idempotent mathematics" or "Tropical mathematics", which are different names for the same thing, is the idea of "idempotent semimodule", also called Maslov's semimodules, the axioms of which are very much like those of vector spaces, or modules over a ring, with two main differences : the field of scalars becomes a semifield, or a semiring, and the "the addition law" of the semimodule, usually denoted by $\oplus$, is idempotent that is, for all elements $x$ of the semimodule in question $x \oplus x = x$; "scalar multiplication" of elements of the semimodule by elements of the semiring, or the semifield, often denoted by $\odot$, behave formally like scalar multiplication in vector spaces. We give only two examples; one is the by now classical Max Plus algebra the other is the Max Times algebra, which we will call here $\mathbb{B}$-algebra.
Max-Plus : Scalars are elements of $\mathbb{F} = \mathbb{R} \cup \{-\infty\}$, the semimodule is $M = (\mathbb{R} \cup \{-\infty\})^n$ ; addition in $\mathbb{F}$ is $s \oplus t = \max\{s, t\}$ and multiplication in $s \odot t = s + t$; in $M$, addition is done coordinatewise as well as multiplication by scalars, that is $(x_1, \cdots, x_n) \oplus (y_1, \cdots, y_n) = (x_1 \oplus y_1, \cdots, x_n \oplus y_n) = (\max\{x_1, y_1\}, \cdots, \max\{x_n, y_n\})$ and $s \odot (x_1, \cdots, x_n) = (x_1 \odot s, \cdots, x_n \odot s) = (x_1 + s, \cdots, x_n + s)$.
$\mathbb{B}$ : $\mathbb{F} = \mathbb{R}_+$, $M = (\mathbb{R}_+)^n$, $x \oplus y = \max\{x, y\}$ and $x \odot y = xy$.
Algebraically, Max Plus and Max Times are isomorphic (as idempotent semimodules); topologically, the structures restricted, respectively to $\mathbb{R}^n$ and $\mathbb{R}^n_{++}$ (the interior of the positive cone $\mathbb{R}^n_+$) are the same (as topological semimodules).
Idempotent mathematics as found applications from operation research (where it originated) to algebraic geometry and mathematical physics. The reader is referred to [20], [23], [27], [29], [31] for more details.
Many "classical concepts and results" have analoguous "tropical versions"; in finite dimension there is a heuristic "tropicalization procedure" leading from the "straight