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# Monotone and weakly confluent set-valued functions and their inverse limits

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ABSTRACT

We present definitions for monotonicity and weak confluence for upper semi-continuous, set-valued functions that generalize those definitions for continuous, single-valued functions. We demonstrate that if the bonding functions of an inverse sequence are monotone (weakly confluent), then various projection maps from the inverse limit will be monotone (weakly confluent) as well.

We use this to show two main results for inverse limits on $[0,1]$. First, if $f_j\colon [0,1] \to 2^{[0,1]}$ is monotone for each $j \in \mathbb{N}$, then $\varprojlim f_j$ is locally connected. Second, if $f_j\colon [0,1] \to 2^{[0,1]}$ is weakly confluent for each $j \in \mathbb{N}$, we establish a sufficient condition for $\varprojlim f_j$ to contain an indecomposable continuum.



## 1. Introduction

Inverse limits have been studied for decades as a mechanism for exploring the properties of continua. In the traditional context, an inverse limit is constructed from a sequence of topological spaces and a sequence of continuous functions between those spaces. In 2004, Mahavier introduced the topic of inverse limits of upper semi-continuous, set-valued functions, and in 2006, this idea was further developed by Ingram and Mahavier [6,9]. Much of the research since then has focused on exploring results from the traditional context of continuous, single-valued functions and determining whether those results can be generalized to the context of upper semi-continuous, set-valued functions.

Capel, [2], presents a number of results concerning inverse limits with monotone bonding functions. He shows that if each bonding function is monotone and either each factor space is an arc or each factor space is a simple closed curve, then the inverse limit is an arc or a simple closed curve respectively. Ingram gives an example, [4, Example 2.4], of a monotone set-valued function on $[0,1]$ whose inverse limit is one-dimensional

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but not an arc. By identifying 0 and 1 to create a circle, this same example illustrates that a monotone set-valued function on a simple closed curve need not have a simple closed curve as its inverse limit either.

Capel also shows that if every factor space is locally connected and every bonding function is monotone, then the inverse limit is locally connected. We present a generalization of this result by showing that if each factor space is an arc and every bonding function is a monotone, set-valued function, then the inverse limit is locally connected.

We also discuss another special class of functions: weakly confluent functions. We give a definition for weakly confluent, set-valued functions and we show that if the bonding functions of an inverse sequence are either all weakly confluent or all monotone, then so are the projection maps defined on the inverse limit and inverse graphs of that sequence.

## 2. Definitions and notation

A set $X$ is a *continuum* if it is a non-empty, compact, connected, Hausdorff space. A continuum which is a subset of a space $X$ is called a *subcontinuum* of $X$.

Given a topological space $X$, we define the following *hyperspaces* of $X$,

$$2^X = \{A \subseteq X \colon A \text{ is non-empty and closed in } X\}$$
$$C(X) = \left\{A \in 2^X \colon A \text{ is connected}\right\}.$$

If $X$ is a metric space with metric $d$, then we define the *Hausdorff metric*, $\mathcal{H}_d$ on $2^X$ by

$$\mathcal{H}_d(E,F) = \inf\left\{\epsilon > 0 \colon E \subseteq \bigcup_{x\in F} B(x,\epsilon) \text{ and } F \subseteq \bigcup_{x\in E} B(x,\epsilon)\right\}$$

where $E, F \in 2^X$ and $B(x,\epsilon)$ represents the open ball in $X$, centered at $x$, with radius $\epsilon$. If the metric space $(X,d)$ is compact, then the metric spaces $(2^X, \mathcal{H}_d)$ and $(C(X), \mathcal{H}_d)$ are compact [10, Theorems 4.13 & 4.17]. In particular, if $X$ is compact, then every sequence of subcontinua of $X$ has a subsequence which converges to a subcontinuum of $X$.

If $X$ and $Y$ are topological spaces, a function $f\colon X \to 2^Y$ is called *upper semi-continuous* if for every $x_0 \in X$ and every open set $V \subseteq Y$ with $f(x_0) \subseteq V$, the set $\{x \in X \colon f(x) \subseteq V\}$ is open in $X$. The *graph* of a function $f\colon X \to 2^Y$ is the set $G(f) = \{(x,y)\colon y \in f(x)\}$. If $X$ and $Y$ are compact Hausdorff spaces, then $f\colon X \to 2^Y$ is upper semi-continuous if and only if $G(f)$ is closed in $X \times Y$ [6, Theorem 2.1]. If $X$ is a $T_1$ space, then every function $f\colon X \to Y$ induces a set-valued function $\tilde{f}\colon X \to 2^Y$ where $\tilde{f}(x) = \{f(x)\}$. In this case, $\tilde{f}$ is upper semi-continuous if and only if $f$ is continuous. In this way, upper semi-continuous, set-valued functions are a generalization of continuous, single-valued functions.

An upper semi-continuous function $f\colon X \to 2^Y$ is called *surjective* if for all $y \in Y$, there is an $x \in X$ with $y \in f(x)$. If $f\colon X \to 2^Y$ and $g\colon Y \to 2^Z$ are upper semi-continuous functions, we define the *composition* $g \circ f\colon X \to 2^Z$ by

$$g \circ f(x) = \bigcup_{y\in f(x)} g(y).$$

If $X$ and $Y$ are compact Hausdorff spaces and $f\colon X \to 2^Y$ is a surjective, upper semi-continuous function, then we define $f^{-1}\colon Y \to 2^X$ by $f^{-1}(y) = \{x\colon y \in f(x)\}$. The graph of $f$ is homeomorphic to the graph of $f^{-1}$, so if $f$ is upper semi-continuous, so is $f^{-1}$.

For each $j \in \mathbb{N}$, let $X_j$ be a topological space, and let $f_j\colon X_{j+1} \to 2^{X_j}$ be upper semi-continuous. Then the sequence of pairs $\{X_j, f_j\}_{j=1}^{\infty}$ is called an *inverse sequence*. The *inverse limit* of the inverse sequence is the space