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Spaces which are retracts or cofactors of paratopological groups ☆

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ABSTRACT

In this paper we investigate Tychonoff spaces which are retracts of paratopological groups. A strong necessary condition for that is the existence of a certain binary operation on the space (called a τ-twister), which was introduced in [2,3]. Some general theorems are established which imply that $\beta\omega$ is not a retract of a paratopological group. We also notice, using some deep results of V.V. Uspenskij, that the space ω_1 of countable ordinals is not a retract of any topological group (see Fact 3).

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1. Introduction

It is well-known that the level of homogeneity of a topological space X can be greatly improved under the product operation. In particular, Jan van Mill has constructed a rigid compact space X such that its square $X \times X$ is homogeneous [9]. This amazing result is well complemented by a remarkable theorem of V.V. Uspenskij who has established that for every Tychonoff space X there exists a Tychonoff space Y such that the product space $X \times Y$ is homogeneous [13]. With this result in mind, it is natural to ask the following questions:

A) Given a Tychonoff space X, is it possible to find a Tychonoff space Y such that $X \times Y$ is homeomorphic to some topological group?

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B) Suppose that X and Y are topological spaces such that the product space $X \times Y$ is homeomorphic to a topological group G. Then how are related the properties of X, Y, and G?

An excursion to the spiritual garden called "General Theory of Topological Spaces" allows to quickly establish some remarkable facts pertinent to questions A) and B).

Fact 1. The space $\beta\omega \setminus \omega$ is not a retract of any topological group, and hence, for no space Y the product space $(\beta\omega \setminus \omega) \times Y$ is homeomorphic to a topological group.

Indeed, otherwise $\beta\omega \setminus \omega$ would have been a Dugundji compactum by a remarkable theorem of V.V. Uspenskij, who has shown that every compact retract of a topological group is a Dugundji compactum (see Theorem 1 in [15]). However, this is not the case, since there are no non-trivial convergent sequences in $\beta\omega\setminus\omega$. It is well-known that every Dugundji compactum with countable tightness is metrizable. Therefore, the next fact also holds:

Fact 2. If X is a non-metrizable compactum with countable tightness, then X is not a retract of any topological group.

In particular, the Alexandroff compactification $A(\omega_1)$ of the discrete space of the cardinality ω_1 is not a retract of any topological group G. It follows that $A(\omega_1) \times Y$ is not homeomorphic to a topological group, for any topological space Y. Some more general statements will be established in Section 4 (see Corollary 4.10). Another idea is used to establish the next fact:

Fact 3. The space ω_1 is not a retract of any topological group, in particular, there is no space Y such that $\omega_1 \times Y$ is homeomorphic to some topological group.

Indeed, otherwise the Stone–Čech compactification $\beta(\omega_1)$ would have been a Dugundji compactum, by a theorem of V.V. Uspenskij in [14]. This would imply that $\omega_1 + 1$ is a Dugundji compactum. Since this conclusion is false, Fact 3 is established.

Thus, there is some interesting information on retracts of topological groups based on deep results of V.V. Uspenskij (some of which were also independently obtained by M.M. Choban, see [15,14,5–7]). See also the article [10] in which the arguments of Uspenskij in [13] were considerably modified and some very interesting special versions of his fundamental theorem were obtained.

However, we could find practically no information on the structure of retracts of paratopological groups which are not topological groups. The first steps in this direction are made in this article. Here is the main general question we are investigating:

Problem 1.1. When is a Tychonoff space X a retract of a paratopological group? a) Find necessary conditions for that. b) Find sufficient conditions for that.

We give some concrete partial answers to this question and questions a) and b). In Section 3 we present some observations related to Questions a) and b) which in a standard way follow from classical results of topological algebra. The main new theorems of this paper are contained in Section 4. They are based on the technique of twisters which was developed by the author in [2] and [3]. The existence of a twister on a space is shown to be a necessary condition for this space to be a retract of a paratopological group. Using this fact, we establish that if a compact space X with countable tightness is a retract of a paratopological group, then X is first-countable (Corollary 4.6). This result provides us with many examples of compact spaces X such that $X \times Y$ is never homeomorphic to a paratopological group. In particular, we see that the Alexandroff compactification of an uncountable discrete space is not a retract of any paratopological

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