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Nested defining sequences and 1 dimensionality of decomposition elements

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ABSTRACT

It is shown that every cell-like, upper semicontinuous decomposition of an n-manifold, $n \neq 4$, that arises from a nested defining sequence can be replaced by a new cell-like decomposition, with the same quotient space and with elements of dimension at most one.

This paper addresses the question of whether a cell-like map from an n-manifold to a finite dimensional space can be approximated by another cell-like map having 1-dimensional point preimages. The normal form for decompositions due to Cannon [2] and Quinn [9] (or see [4, Theorem 40.6]) assures this can be done when $n \geq 5$ and X is an n-manifold except for a singular set $S(X)$ of dimension at most $n-3$. The same conclusion also holds when $S(X)$ is contained in an $(n-1)$-manifold. However, the general case remains unresolved.

Here we settle a prominent case in which the map arises as the decomposition map associated with a nested defining sequence in the domain M. The term is explained in the next section. The following is the central result.

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Main Theorem 0.1. *Let M be a compact n-manifold, $n \neq 4$, and G a cell-like upper semicontinuous decomposition of M arising from a nested defining sequence. Then, for each $\epsilon > 0$ the decomposition map $\pi : M \to X = M/G$ can be ϵ-approximated by a cell-like map $F : M \to X$ such that each point preimage $F^{-1}(x)$, $x \in X$, has dimension at most 1.*

1. Definitions and basic properties

Let M be an n-manifold. A *nested defining sequence* for M is a sequence $\mathscr{S} = \{\mathscr{M}_1, \mathscr{M}_2, \ldots\}$ satisfying the following three conditions:

Disjointness Criterion. For each index i the set $\mathscr{M}_i$ is a locally finite collection $\{A_j\}$ of compact n-manifolds with boundary with pairwise disjoint interiors.

Nested Criterion. For each index $i > 1$, each $A \in \mathscr{M}_i$ has a unique predecessor, Pre$A$, in $\mathscr{M}_{i-1}$ that properly contains A.

Boundary Size Criterion. For each index i, each $A \in \mathscr{M}_i$ and each pair of distinct points $x, y \in \partial A$ there is an integer $s > i$ such that no element of $\mathscr{M}_s$ contains both x and y.

These conditions are slightly more general than those of [3] or [4, Section 34], but they are all that are essential.

The decomposition G of M arising from the nested defining sequence $\mathscr{S} = \{\mathscr{M}_1, \mathscr{M}_2, \ldots\}$ is the relation given by the rule that $x, y \in M$ belong to the same element of G if there exists a fixed integer k, depending simply on x and y, such that each $\mathscr{M}_i$ has a chain of at most k elements connecting x to y; explicitly, there is a chain $A_1, A_2, \ldots, A_k$ of elements from $\mathscr{M}_i$ such that $x \in A_1$, $y \in A_k$ and $A_j \cap A_{j+1} \neq \emptyset$ for $j = 1, 2, \ldots, k-1$. It should be clear that this is an equivalence relation. It will follow from Lemma 1.1 below that equivalent points x, y are actually connected by a chain of length two, but that is not required by the definition.

Since the focus here is on cell-like decompositions, we insist that the nested defining sequence also satisfies the following, which assures that the decomposition arising from the sequence is cell-like; see [3, Theorem 1] or [4, Prop. 34.4].

Null Homotopy Criterion. For $i > 1$ and any $A \in \mathscr{M}_i$ the inclusion $A \to PreA$ is homotopic to a constant.

Let X be a space and $\mathscr{M}$ a collection of subsets of X. Given $Z \subset X$, define its *star with respect to* $\mathscr{M}$ as

$$St(Z, \mathscr{M}) = Z \cup \{M \in \mathscr{M} | M \cap Z \neq \emptyset\}.$$

Set $St^1(Z, \mathscr{M}) = St(Z, \mathscr{M})$ and recursively for each integer $k > 1$ set

$$St^k(Z, \mathscr{M}) = St(St^{k-1}(Z, \mathscr{M}), \mathscr{M}).$$

The following lemma ([3, Addendum to Theorem 1] or [4, Lemma 34.1]) presents important information about the decomposition G.

Lemma 1.1.

(a) For each $x \in X$, $\pi^{-1}\pi(x) = \cap_i St^2(x, \mathscr{M}_i)$.
(b) No element $g \in G$ contains more than one point of the set $\partial\mathscr{S}$ defined as $\cup_i\{\partial A | A \in \mathscr{M}_i\}$.
(c) If $x \in g \in G$ and if either $x \in \partial\mathscr{S}$ or $g \cap \partial\mathscr{S} = \emptyset$, then $g = \pi^{-1}\pi(x) = \cap_i St(x, \mathscr{M}_i)$.

Decompositions arising from a nested defining sequence can be rather complex. After all, the concept was introduced to define the Totally Wild Flow on $S^n \times \mathbb{R}$, which involves a cell-like decomposition of S^n, all elements of which are non-cellular, thereby implying that no point of the decomposition space has a manifold