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Topological representation of precontact algebras and a connected version of the Stone Duality Theorem – I

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ABSTRACT

The notions of a *2-precontact space* and a *2-contact space* are introduced. Using them, new representation theorems for precontact and contact algebras are proved. They incorporate and strengthen both the discrete and topological representation theorems from [5,6,8,9,21]. It is shown that there are bijective correspondences between such kinds of algebras and such kinds of spaces. As applications of the obtained results, we get new connected versions of the Stone Duality Theorems [19] for Boolean algebras and for complete Boolean algebras, as well as a Smirnov-type theorem (in the sense of [17]) for a kind of compact T_0-extensions of compact Hausdorff extremally disconnected spaces. We also introduce the notion of a *Stone adjacency space* and using it, we prove another representation theorem for precontact algebras. We even obtain a bijective correspondence between the class of all, up to isomorphism, precontact algebras and the class of all, up to isomorphism, Stone adjacency spaces.

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1. Introduction

In this paper we give the proofs of the results announced in the first seven sections of our paper [7] (the results of the remaining sections will be proved in the second part of the paper) and obtain many new additional results and some new applications. In it we present a common approach both to the discrete and to the non-discrete region-based theory of space. The paper is a continuation of the investigations started in [21] and continued in [5,6,9].

Standard models of non-discrete theories of space are the contact algebras of regular closed subsets of some topological spaces [5,6,9,21,16,18]. In a sense these topological models reflect the continuous nature of the space. However, in the "real-world" applications, where digital methods of modeling are used, the continuous models of space are not enough. This motivates a search for good "discrete" versions of the theory of space. One kind of discrete models are the so called *adjacency spaces*, introduced by Galton [12] and generalized by Düntsch and Vakarelov in [8]. Based on the Galton's approach, Li and Ying [14] presented a "discrete" generalization of the Region Connection Calculus (RCC). The latter, introduced in [15], is one of the main systems in the non-discrete region-based theory of space. A natural class of Boolean algebras related to adjacency spaces are the *precontact algebras*, introduced in [8] under the name of *proximity algebras*. The notion of precontact algebra is a generalization of the notion of contact algebra. Each adjacency space generates canonically a precontact algebra. It is proved in [8] (using another terminology) that each precontact algebra can be embedded in the precontact algebra of an adjacency space. In [5] we proved that each contact algebra can be embedded in the standard contact algebra of a compact semiregular T_0-space, answering the question of Düntsch and Winter, posed in [9], whether the contact algebras have a topological representation. This shows that contact algebras possess both a discrete and a non-discrete (topological) representation. In this paper we extend the representation techniques developed in [5,6] to precontact algebras, proving that each precontact algebra can be embedded in a special topological object, called a *2-precontact space*. We also establish a bijective correspondence between all, up to isomorphism, precontact algebras and all, up to isomorphism, 2-precontact spaces. This result is new even in the special case of contact algebras: introducing the notion of *2-contact space* as a specialization of the notion of a 2-precontact space, we show that there is a bijective correspondence between all, up to isomorphism, contact algebras and all, up to isomorphism, 2-contact spaces. Also, we introduce the notion of a *Stone adjacency space* and using it, we prove another representation theorem for precontact algebras. We even obtain a bijective correspondence between the class of all, up to isomorphism, precontact algebras and the class of all, up to isomorphism, Stone adjacency spaces.

The developed theory permits us to obtain as corollaries the celebrated Stone Representation Theorem [19] and a new connected version of it. They correspond, respectively, to the extremal contact relations on Boolean algebras: the smallest one and the largest one. We show as well that the new connected version of the Stone Representation Theorem can be extended to a new connected version of the Stone Duality Theorem. Let us explain what we mean by a "connected version". The celebrated Stone Duality Theorem [19] states that the category **Bool** of all Boolean algebras and Boolean homomorphisms is dually equivalent to the category **Stone** of compact Hausdorff totally disconnected spaces (i.e., *Stone spaces*) and continuous maps. The restriction of the Stone duality to the category **CBool** of complete Boolean algebras and Boolean homomorphisms is a duality between the category **CBool** and the category of compact Hausdorff extremally disconnected spaces and continuous maps. We introduce the notion of a *Stone 2-space* and the category **2Stone** of Stone 2-spaces and suitable morphisms between them, and we show that the category **2Stone** is dually equivalent to the category **Bool**. The Stone 2-spaces are pairs (X, X_0) of a compact *connected* T_0-space X and a dense subspace X_0 of X, satisfying some mild conditions. We introduce as well the notion of an *extremally connected space* and show that the category **ECS** of extremally connected spaces and continuous maps between them satisfying a natural condition, is dually equivalent to the category **CBool**.