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Remarks on straight finite decomposition complexity

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In their previous paper [7], the authors introduced the straight (non-game-theoretic) counterpart of the finite decomposition complexity defined by E. Guentner, R. Tessera, G. Yu. In the present paper, we correct a proof of a statement from [7] that the straight finite decomposition complexity implies Property A.
We also discuss a dimensional-like ordinal-valued invariant related to the straight finite decomposition complexity.

1. Introduction

The notion of decomposition complexity was introduced in [13] using a game theoretical approach. In the paper [7], the authors introduced the notion of straight decomposition complexity. Since then, the class of spaces having straight finite decomposition complexity (sFDC) was considered by different authors; see, e.g., [10,1,16,8,9]. In particular, the spaces with sFDC found applications in the algebraic K-theory [10].

Recently, it turned out that the proof of the implication "sFDC ⇒ property A" from [7] contains a gap. In Section 4 we present a corrected proof. Note that J. Dydak [8] found another proof of the above implication.

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We also apply a stratification procedure to the class of spaces having sFDC in order to obtain a dimension-like asymptotic invariant (Section 3). Some permanence results (see [12]) for this invariant are discussed.

We are thankful to Takamitsu Yamauchi for spotting a gap in our paper [7].

2. Preliminaries

All spaces under consideration are metrizable. If (X, d) is a metric space and A, B are nonempty subsets of X, we let $d(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}$. Given $R > 0$, we say that a family $\mathcal{A}$ of nonempty subsets of X is R-disjoint if $d(A, B) > R$, for every $A, B \in \mathcal{A}$.

A metric space X is geodesic if for every $x, y \in X$ there exists an isometric embedding $\alpha\colon [0, d(x, y)] \to X$ such that $\alpha(0) = x$ and $\alpha(d(x, y)) = y$.

Let $\mathcal{X}, \mathcal{Y}$ be families of metric spaces and $R > 0$. We say that $\mathcal{X}$ is R-decomposable over $\mathcal{Y}$ if, for any $X \in \mathcal{X}$, $X = \bigcup(\mathcal{V}_1 \cup \mathcal{V}_2)$, where $\mathcal{V}_1, \mathcal{V}_2$ are R-disjoint families and $\mathcal{V}_1 \cup \mathcal{V}_2 \subset \mathcal{Y}$.

A family $\mathcal{X}$ of metric spaces is said to be *bounded* if

$$\mathrm{mesh}(\mathcal{X}) = \sup\{\mathrm{diam}\, X \mid X \in \mathcal{X}\} < \infty.$$

Let $\mathfrak{A}$ be a collection of metric families. A metric family $\mathcal{X}$ is *decomposable over* $\mathfrak{A}$ if, for every $r > 0$, there exists a metric family $\mathcal{Y} \in \mathfrak{A}$ and an r-decomposition of $\mathcal{X}$ over $\mathcal{Y}$.

The following notion is introduced in [13].

2.1. Definition. We consider the metric decomposition game of two players, a defender and a challenger. Let $\mathcal{X} = \mathcal{Y}_0$ be the starting family. On the first turn the challenger asserts $R_1 > 0$, the defender responds by exhibiting an R_1-decomposition of $\mathcal{Y}_0$ over a new metric family $\mathcal{Y}_1$. On the second turn, the challenger asserts an integer R_2, the defender responds by exhibiting an R_2-decomposition of $\mathcal{Y}_1$ over a new metric family $\mathcal{Y}_2$. The game continues in this way, turn after turn, and ends if and when the defender produces a bounded family. In this case the defender has won.

A metric family $\mathcal{X}$ has FDC if the defender has always a winning strategy. A metric space X has FDC if the family $\{X\}$ does.

2.2. Definition. We say that a metric space X satisfies the *straight Finite Decomposition Property* (sFDC) if, for any sequence $R_1 < R_2 < \dots$ of positive numbers, there exists $n \in \mathbb{N}$ and metric families $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_n$ such that $\{X\}$ is R_1-decomposable over $\mathcal{V}_1$, $\mathcal{V}_i$ is R_i-decomposable over $\mathcal{V}_{i+1}$, $i = 1, \dots, n-1$, and the family $\mathcal{V}_n$ is bounded.

From the definition it easily follows that any space that has the FDC also has the straight FDC.

We recall that the *asymptotic dimension* of a metric space does not exceed n, $\mathrm{asdim}\, X \le n$ if for every $R > 0$ there are uniformly bounded R-disjoint families $\mathcal{U}_i$, $i = 0, \dots, n$ of sets in X such that the family $\cup_{i=1}^n \mathcal{U}_i$ is a cover of X [11].

The following notion was introduced in [5].

2.3. Definition. A metric space X is said to have the *asymptotic property C* if for every sequence $R_1 < R_2 < \dots$ there exists $n \in \mathbb{N}$ and uniformly bounded R_i-disjoint families $\mathcal{U}_i$, $i = 1, \dots, n$, such that the family $\cup_{i=1}^n \mathcal{U}_i$ is a cover of X.

A map $f\colon X \to Y$ of metric spaces is a coarse embedding if there exist unbounded increasing functions $\varphi, \psi\colon [0, \infty) \to [0, \infty)$ such that