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Companions of directed sets and the Ordering Lemma ☆

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Dedicated to the memory of Alex Chigogidze

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ABSTRACT

Given a partially ordered set $(D, \leq)$, a companion $(C \preceq)$ of $(D, \leq)$ is a well ordered set where C is a cofinal subsets of $(D, \leq)$ such that for every $c_1, c_2 \in C$ if $c_1 \leq c_2$ then $c_1 \preceq c_2$. The Ordering Lemma says that every partially ordered set has a companion. Given a directed set $(D, \leq)$ and a net $f : D \to X$, the restriction $f \restriction C$ of the net to the companion $(C, \preceq)$ of $(D, \leq)$ is a transfinite sequence. We show how the convergence and clustering of $f \restriction C$ is related to the convergence and clustering of f.

1. Introduction

This paper concerns partially ordered sets, directed sets, nets, transfinite sequences, and the Ordering Lemma. The Ordering Lemma is a version of the Axiom of Choice that was introduced by Norman Howes in his dissertation [2] and popularized in his book "Modern Analysis and Topology" [5][1]

☆ Some of the results in this paper were presented at the 30th Summer Conference on Topology and its Applications, NUI, Galway, Ireland [17], and the First Pan Pacific International Conference on Topology and its Applications, Minnan University, Zhangzhou, China [18].

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[1] The statement of the Ordering Lemma in Howes' book [5], is not completely clear as pointed out by John Mack [10]. However, the Ordering Lemma is also given in [3,4] and [14].

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We state the Ordering Lemma using the following definition.

Definition 1.1. Let $(D, \leq)$ be a partially ordered set. We call a well ordered set $(C, \preceq)$ *a companion of* $(D, \leq)$ provided C is a cofinal subset of D and $\preceq$ is a well order on C that is compatible with the partial order $\leq$ on D i.e., for every $c_1, c_2 \in C$ if $c_1 \leq c_2$ then $c_1 \preceq c_2$.

With this definition the Ordering Lemma has the simple statement:

Lemma 1.2. *([2, Theorem 1.1]) Every partially ordered set has a companion.*

Recall that $(D, \leq)$ is called a *directed set* provided $(D, \leq)$ is a partially ordered set in which every finite subset has an upper bound, and is called ω-directed if every countable subset has an upper bound. A net is a function whose domain is a directed set. A net is called a *transfinite sequence* if its domain is a well ordered set (further definitions are given in §2).

Given a net

$$f : D \to X$$

into a topological space X and any companion $(C, \preceq)$ of $(D, \leq)$, the restriction of the net

$$f \upharpoonright C : C \to X$$

is a transfinite sequence with respect to the well order $\preceq$ on C rather than the partial order on C inherited from $(D, \leq)$. We call $f \upharpoonright C$ the *companion (transfinite) sequence of the net* f.

It is natural to ask the question: What bearing does the convergence or clustering of a net f, or its companion sequence $f \upharpoonright C$, have on the other one? We show in Lemma 4.2 that if either f or $f \upharpoonright C$ converges then the other one clusters (half of this is due to Howes [2]). In §4 we show that no other implications of this nature hold in general. Nevertheless, the following question is especially interesting.

Question 1.3. *If a companion sequence $f \upharpoonright C$ has a cluster point, does the net f have a cluster point?*

We show that Question 1.3 does not hold in all cases. This question arose from a preprint of W. Sconyers and N. Howes [14, Theorem 4] where they implicitly assume a positive answer to Question 1.3 for ω-directed sets. Assuming a positive answer to Question 1.3, allowed them to claim that every normal linearly Lindelöf space is Lindelöf, and that would answer a well known question raised in 1968 (cf. [15]). In this paper we provide results (see Example 4.6 and Theorem 1.4 (2)) that give a negative answer to Question 1.3 for certain ω-directed sets. Moreover, the ω-directed sets in our Example 4.6 not only have a companion which answers Question 1.3 in the negative but also have another companion that answers Question 1.3 in the positive. Thus the choice of companion can be involved in answering Question 1.3. We show that such examples hinge on whether or not the directed set has a well ordered cofinal subset. Our main result is

Theorem 1.4.
(1) *If $(D, \leq)$ has a well ordered cofinal subset then there exists a companion $(C, \preceq)$ of $(D, \leq)$ such that for every net $f : D \to X$, if $f \upharpoonright C$ clusters at $x \in X$, then the net f clusters at x.*
(2) *If $(D, \leq)$ does not have a well ordered cofinal subset then there exist a companion $(C, \preceq)$ of $(D, \leq)$ and a net $f : D \to X$ such that the companion sequence $f \upharpoonright C$ has a cluster point, but the net f does not have a cluster point.*

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