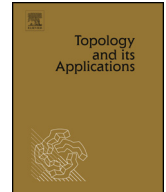




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Topology and its Applications

www.elsevier.com/locate/topolCompanions of directed sets and the Ordering Lemma [☆]

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Dedicated to the memory
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ABSTRACT

Given a partially ordered set (D, \leq) , a companion (C, \preceq) of (D, \leq) is a well ordered set where C is a cofinal subsets of (D, \leq) such that for every $c_1, c_2 \in C$ if $c_1 \leq c_2$ then $c_1 \preceq c_2$. The Ordering Lemma says that every partially ordered set has a companion. Given a directed set (D, \leq) and a net $f : D \rightarrow X$, the restriction $f \upharpoonright C$ of the net to the companion (C, \preceq) of (D, \leq) is a transfinite sequence. We show how the convergence and clustering of $f \upharpoonright C$ is related to the convergence and clustering of f .

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1. Introduction

This paper concerns partially ordered sets, directed sets, nets, transfinite sequences, and the Ordering Lemma. The Ordering Lemma is a version of the Axiom of Choice that was introduced by Norman Howes in his dissertation [2] and popularized in his book “Modern Analysis and Topology” [5]¹

[☆] Some of the results in this paper were presented at the 30th Summer Conference on Topology and its Applications, NUI, Galway, Ireland [17], and the First Pan Pacific International Conference on Topology and its Applications, Minnan University, Zhangzhou, China [18].

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¹ The statement of the Ordering Lemma in Howes’ book [5], is not completely clear as pointed out by John Mack [10]. However, the Ordering Lemma is also given in [3,4] and [14].

We state the Ordering Lemma using the following definition.

Definition 1.1. Let (D, \leq) be a partially ordered set. We call a well ordered set (C, \preceq) a *companion* of (D, \leq) provided C is a cofinal subset of D and \preceq is a well order on C that is compatible with the partial order \leq on D i.e., for every $c_1, c_2 \in C$ if $c_1 \leq c_2$ then $c_1 \preceq c_2$.

With this definition the Ordering Lemma has the simple statement:

Lemma 1.2. (*[2, Theorem 1.1]*) *Every partially ordered set has a companion.*

Recall that (D, \leq) is called a *directed set* provided (D, \leq) is a partially ordered set in which every finite subset has an upper bound, and is called ω -directed if every countable subset has an upper bound. A net is a function whose domain is a directed set. A net is called a *transfinite sequence* if its domain is a well ordered set (further definitions are given in §2).

Given a net

$$f : D \rightarrow X$$

into a topological space X and any companion (C, \preceq) of (D, \leq) , the restriction of the net

$$f \upharpoonright C : C \rightarrow X$$

is a transfinite sequence with respect to the well order \preceq on C rather than the partial order on C inherited from (D, \leq) . We call $f \upharpoonright C$ the *companion (transfinite) sequence of the net f* .

It is natural to ask the question: What bearing does the convergence or clustering of a net f , or its companion sequence $f \upharpoonright C$, have on the other one? We show in [Lemma 4.2](#) that if either f or $f \upharpoonright C$ converges then the other one clusters (half of this is due to Howes [\[2\]](#)). In §4 we show that no other implications of this nature hold in general. Nevertheless, the following question is especially interesting.

Question 1.3. *If a companion sequence $f \upharpoonright C$ has a cluster point, does the net f have a cluster point?*

We show that [Question 1.3](#) does not hold in all cases. This question arose from a preprint of W. Sconyers and N. Howes [\[14, Theorem 4\]](#) where they implicitly assume a positive answer to [Question 1.3](#) for ω -directed sets. Assuming a positive answer to [Question 1.3](#), allowed them to claim that every normal linearly Lindelöf space is Lindelöf, and that would answer a well known question raised in 1968 (cf. [\[15\]](#)). In this paper we provide results (see [Example 4.6](#) and [Theorem 1.4](#) (2)) that give a negative answer to [Question 1.3](#) for certain ω -directed sets. Moreover, the ω -directed sets in our [Example 4.6](#) not only have a companion which answers [Question 1.3](#) in the negative but also have another companion that answers [Question 1.3](#) in the positive. Thus the choice of companion can be involved in answering [Question 1.3](#). We show that such examples hinge on whether or not the directed set has a well ordered cofinal subset. Our main result is

Theorem 1.4.

- (1) *If (D, \leq) has a well ordered cofinal subset then there exists a companion (C, \preceq) of (D, \leq) such that for every net $f : D \rightarrow X$, if $f \upharpoonright C$ clusters at $x \in X$, then the net f clusters at x .*
- (2) *If (D, \leq) does not have a well ordered cofinal subset then there exist a companion (C, \preceq) of (D, \leq) and a net $f : D \rightarrow X$ such that the companion sequence $f \upharpoonright C$ has a cluster point, but the net f does not have a cluster point.*

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