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Pairwise k-semi-stratifiable bispaces and topological ordered spaces ☆

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ABSTRACT

In this paper, we continue to study pairwise (k-semi-)stratifiable bitopological spaces. Some new characterizations of pairwise k-semi-stratifiable bitopological spaces are provided. Relationships between pairwise stratifiable and pairwise k-semi-stratifiable bitopological spaces are further investigated, and an open question recently posed by Li and Lin in [18] is completely solved. We also study the quasi-pseudo-metrizability of a topological ordered space $(X, \tau, \preccurlyeq)$. It is shown that if $(X, \tau, \preccurlyeq)$ is a ball transitive topological ordered C- and I-space such that τ is metrizable, then its associated bitopological space $(X, \tau^\flat, \tau^\natural)$ is quasi-pseudo-metrizable. This result provides a partial affirmative answer to a problem in [15].

1. Introduction

Undoubtedly, topology and order are not only important topics in mathematics but also applicable in many other disciplines. For example, nonsymmetric notions of distance are needed for mathematical modeling in the natural, physical and cybernetic sciences and the corresponding topological notion is that of a quasi-metric or a quasi-pseudo-metric. The study of quasi-metrizable spaces naturally leads to the concepts of quasi-uniformities and bitopological spaces. In this aspect, Kelly's seminal paper [14] made

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pioneer contributions. On the other hand, the notions of a sober space and the Scott topology, in line with the investigation of partial orders and pre-orders, are useful in theoretical computer science in the study of algorithms which act on other algorithms. Moreover, finite topological spaces (i.e., finite pre-orders) can be used to construct a mathematical model of a video monitor screen which may be useful in computer graphics.

Since Kelly's work in [14], bitopological spaces have attracted the attention of many researchers. For example, Reilly [29] explored separation axioms for bitopological spaces, Cooke and Reilly [4] discussed the relationships between six definitions of bitopological compactness appeared in the literature, Raghavan and Reilly [28] introduced a notion of bitopological paracompactness and established a bitopological version of Michael's classical characterization of regular paracompact spaces. In addition to separation and covering properties, generalized metric properties have also been considered in the setting of bitopological spaces. In this direction, Fox [7] discussed the quasi-metrizability of bitopological spaces, pairwise stratifiable bitopological spaces and their generalizations have been introduced and studied in [12,21,22].

Interplay between topology and order has been a very interesting area. In 1965, Nachbin's book [26] was published. This book is one of general references on the subject available today, and it covers results obtained by the author in his research on spaces with structures of order and topology. Among many topics in this line, McCartan [23] studied bicontinuous (herein called C-space and I-space) pre-ordered topological spaces and investigated the relationships between the topology of such a space and two associated convex topologies, and Faber [6] studied metrizability in generalized ordered spaces. Recently, there have been some renewed interests in the study of generalized metric properties in bitopological spaces and topological ordered spaces. Künzi and Mushaandja investigated the quasi-pseudo-metrizability of a topological ordered space in [15] and [25], respectively. They obtained some results related to the upper topology $\tau^\natural$ and the lower topology $\tau^\flat$ of a metrizable ordered space $(X, \tau, \preccurlyeq)$ which is both a C- and an I-space in the sense of Priestle [27]. Moreover, Li [16] as well as Li and Lin [17] carried on the study of pairwise (semi-)stratifiable bispaces and established some new characterizations for these classes of bitopological spaces. In a very recent paper [18], Li and Lin further introduced and studied the class of k-semi-stratifiable bitopological spaces.

The main purpose of this paper is to continue the study of pairwise k-semi-stratifiable bitopological spaces and their relationships with and applications to the quasi-pseudo-metrizability of topological ordered spaces. For the sake of self-completeness, in Section 2, we introduce the necessary definitions and terminologies. In Section 3, we provide some new characterizations of pairwise k-semi-stratifiable bitopological spaces in terms of g-functions, σ-cushioned pair k-networks and cs-networks. In Section 4, we consider some conditions under which a pairwise k-semi-stratifiable bitopological space is pairwise stratifiable. An open question posed in [18] is completely solved and results in [2] and [3] are extended to the setting of bitopological spaces. In the last section, we consider the quasi-pseudo-metrizability of topological ordered spaces and provide a partial affirmative answer to an open problem of Künzi and Mushaandja in [15].

Our notations in this paper are standard. For any undefined concepts and terminologies, we refer the reader to [5] or [10].

2. Preliminaries and notations

A *quasi-pseudo-metric* d on a nonempty set X is a non-negative real-valued function $d : X \times X \to \mathbb{R}_+$ such that (i) $d(x, x) = 0$ and (ii) $d(x, z) \le d(x, y) + d(y, z)$, for all $x, y, z \in X$. If d is a quasi-pseudo-metric on X, then the ordered pair (X, d) is called a *quasi-pseudo-metric space*. Every quasi-pseudo-metric d on X induces a topology $\tau(d)$ on X which has as a base the family $\{B_d(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. Every quasi-pseudo-metric d on X induces a conjugate quasi-pseudo-metric d^{-1} on X, defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$. A *bitopological space* [14] (for short, *bispace* [12]) is a triple (X, τ_1, τ_2), where X is a nonempty set, topologies τ_1 and τ_2 are two topologies on X. A bispace