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# The 6- and 8-palette numbers of links ☆

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ABSTRACT

For an effectively $n$-colorable link $L$, $\mathrm{C}^*_n(L)$ stands for the minimum number of distinct colors used over all effective $n$-colorings of $L$. It is known that $\mathrm{C}^*_n(L) \geq 1 + \log_2 n$ for any effectively $n$-colorable link $L$ with non-zero determinant. The aim of this paper is to prove that $\mathrm{C}^*_6(L) = 4$ and $\mathrm{C}^*_8(L) = 5$ for any effectively 6- and 8-colorable link $L$, respectively. For ribbon 2-links, we prove the same equalities for $n = 6$ and 8, and $\mathrm{C}^*_{13}(L) = 5$ for $n = 13$.



## 1. Introduction

Fox $n$-colorings [3] are well-known and fundamental knot invariants, and have been extensively studied. For a given positive integer $n$ and a knot $K$ that has a non-trivial $n$-coloring, the minimum number of colors used among all non-trivially $n$-colored diagrams of $K$ was originally introduced in [5], and has been studied for small numbers $n$ up to $n = 13$, as reviewed below.

Effective $n$-colorings were defined in [8] to study $n$-colorings for composite numbers $n$. An $n$-coloring is *effective* if the $p$-coloring obtained by reduction modulo $p$ is non-trivial for every prime factor $p$ of $n$. For an integer $n \geq 2$, the *$n$-palette number* of an effectively $n$-colorable link in $\mathbb{R}^3$ or a surface-link in $\mathbb{R}^4$ is the

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minimum number of distinct colors used over all effectively $n$-colored diagrams of $L$. We denote by $\mathrm{C}_n^*(L)$ the palette number of an effectively $n$-colorable link $L$. If $n$ is prime, then an $n$-coloring is effective if and only if it is non-trivial. In this case, the palette number is coincident with the minimum number of colors introduced in [5].

The palette number has been determined for small $n$ as follows. By definition, we have $\mathrm{C}_2^*(L) = 2$ for any 2-colorable (surface-)link $L$, and $\mathrm{C}_n^*(L) = 2$ for any splittable (surface-)link $L$ with $n \geq 2$. We also have $\mathrm{C}_3^*(L) = 3$ for any 3-colorable and non-splittable (surface-)link $L$. The first non-trivial result was given for $n = 5$ [18]. It holds that $\mathrm{C}_5^*(K) = 4$ for any 5-colorable knot or ribbon 2-knot $K$, and there is a non-ribbon 2-knot $K$ with $\mathrm{C}_5^*(K) = 5$ [18]. Similarly, for $n = 7$, $\mathrm{C}_7^*(K) = 4$ holds for any 7-colorable knot or ribbon 2-knot $K$ [14], and there is a non-ribbon 2-knot $K$ with $\mathrm{C}_7^*(K) = 6$ [15].

On the other hand, for any $n \geq 4$, there is an effectively $n$-colorable and non-splittable link $L$ with $\mathrm{C}_n^*(L) = 4$, although $\det(L) = 0$ [16]. When we are restricted to an effectively $n$-colorable link with $\det(L) \neq 0$, the lower bound is given by $\mathrm{C}_n^*(L) \geq 1 + \log_2 n$ [6], which is a generalization of the inequality in the case of knots and prime numbers $n$ [11]. For $n = 9$ and 11, it was shown in [12] and [13], respectively, that the equality $\mathrm{C}_n^*(L) = 5$ holds for any effectively $n$-colorable link or ribbon 2-link $L$, and for $n = 13$, the same equality holds for any effectively 13-colorable link in $\mathbb{R}^3$ [1,2].

The aim of this paper is to prove that $\mathrm{C}_6^*(L) = 4$ and $\mathrm{C}_8^*(L) = 5$ (Theorems 2.5 and 2.10), thereby completing the list of values of $\mathrm{C}_n^*(L)$ for $n \leq 9$.

This paper is organized as follows. In Section 2, we present a summary of various results on the $n$-palette number and properties of effectively $n$-colored link diagrams. In Section 3, we prove that any effectively 6-colorable link has a diagram colored by four colors $0, 1, 2$, and 3. In Section 4, we prove that any effectively 8-colorable link has a diagram colored by five colors $0, 1, 2, 3$, and 6. Section 5 is devoted to studying the case of effectively 6-, 8-, or 13-colorable ribbon 2-links.

## 2. Preliminaries

A diagram of a link is regarded as a disjoint union of arcs obtained from its projection image in a plane by cutting it at under-crossings. It may contain embedded circles without under-crossings, and we also regard them as arcs of $D$ for convenience.

For an integer $n \geq 2$ and a diagram $D$ of a link $L$, a *Fox $n$-coloring* [3] (or simply an *$n$-coloring*) for $D$ is a map

$$C : \{\text{the arcs of } D\} \to \mathbb{Z}/n\mathbb{Z}$$

such that the congruence $a + c \equiv 2b \pmod{n}$ holds at every crossing of $D$, where $a$ and $c$ are the elements of $\mathbb{Z}/n\mathbb{Z}$ assigned to the under-arcs by $C$ and $b$ is the element assigned to the over-arc. If an element $a \in \mathbb{Z}/n\mathbb{Z}$ is assigned to an arc of $D$ by $C$, then $a$ is called the *color* of the arc, and the arc is called an *$a$-arc*. The *color* of a crossing is $\{a|b|c\}$ if the under-arcs are $a$- and $c$-arcs and the over-arc is a $b$-arc. We say that the color $\{a|b|c\}$ of a crossing is *trivial* if $a = b = c$, and otherwise *non-trivial*.

An $n$-coloring $C$ for $D$ is called *trivial* if the map $C$ is constant; that is, all the arcs of $D$ receive a single color, and otherwise *non-trivial*. Furthermore, an $n$-coloring $C$ is called *effective* [8] if for any prime factor $p$ of $n$, the $p$-coloring $\pi_p^n \circ C$ is non-trivial, where $\pi_p^n : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ is the natural projection. It follows by the Chinese remainder theorem that, for a composite $n$, an $n$-coloring $C$ is non-trivial if and only if there is a prime factor $p$ of $n$ such that the $p$-coloring $\pi_p^n \circ C$ is non-trivial. Therefore, the effective $n$-colorability is stronger than the non-trivial $n$-colorability provided that $n$ is a composite number.

We say that a link $L$ is *$n$-colorable* if a diagram of $L$ has a non-trivial $n$-coloring, and *effectively $n$-colorable* if a diagram has an effective $n$-coloring. By using Reidemeister moves, we see that this definition does not