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On the spectrum of dynamical systems on trees

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ABSTRACT

In their paper, Schweizer and Smítal (1994) [10] introduced the notions of distributional chaos for continuous maps of the interval, spectrum and weak spectrum of a dynamical system. Among other things , they have proved that in the case of continuous interval maps, both the spectrum and the weak spectrum are finite and generated by points from the basic sets. Here we generalize the mentioned results for the case of continuous maps of a finite tree. While the results are similar, the original argument is not applicable directly and needs essential modifications. In particular, it was necessary to resolve the problem of intersection of basic sets, which was a crucial point.
An example of one-dimensional dynamical system with an infinite spectrum is presented.

1. Introduction and the main results

Let X be a compact metric space and d be its metric. We consider a continuous map $f\colon X \to X$ (we denote the set of such maps by $C(X, X)$), in this setting we talk about a *dynamical system*. Let $\mathbb{N}$ be the set of positive integers, taking an $x \in X$, we define recursively an nth *iteration* by putting $f^0(x) = x$ and $f^n(x) = f(f^{n-1}(x))$, for any $n \in \mathbb{N}$. The sequence $(f^n(x))_{n=0}^{\infty}$ is called a *trajectory of point* x and denoted by $\mathrm{tr}_f(x)$. We define an *orbit of a point* $x \in X$ by putting $\mathrm{Orb}_f(x) = \{f^n(x);\ n \in \mathbb{N} \cup \{0\}\}$, similarly, we define an orbit of a set $A \subset X$ as $\mathrm{Orb}_f(A) = \{f^n(x);\ x \in A \text{ and } n \in \mathbb{N} \cup \{0\}\}$. If we have $f^p(x) = x$ for some $x \in X$ and some positive integer p, then we call the point x *periodic* and the smallest p of such a kind is called the *period of* x.

We say that f is *transitive*, if for every two nonempty open sets U and V in X there is an integer n, such that $f^n(U) \cap V \neq \emptyset$. Let K be a nonempty subset of X, we say that $f|_K$ is *invariant* if $f(K) \subset K$.

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In addition, if $f(K) = K$ holds, then we say that $f|_K$ is *strongly invariant*. Let A be a subset of X, then $\operatorname{diam} A = \sup \{\mathrm{d}(x, y); x, y \in A\}$.

Let $f\colon X \to Y$ be a map between two metric spaces, then we say that f is *monotone* on X if for any $y \in Y$ the set $f^{-1}(y)$ is connected.

Let $f \in C(X, X)$ and $g \in C(Y, Y)$ be maps of compact metric spaces, a continuous surjection $\varphi\colon X \to Y$ is called *semiconjugacy* if $\varphi \circ f = g \circ \varphi$. Moreover, if there exists a $k \in \mathbb{N}$ such that $\#\varphi^{-1}(y) \leq k$ for each $y \in Y$, then we say that φ semiconjugates f and g *almost exactly*.

For a given $x \in X$ we define the *omega-limit set of* x as a set of accumulation points of trajectory $\operatorname{tr}_f(x)$ and we denote it as $\omega_f(x)$. We say that $\omega_f(x)$ is *maximal* if and only if there is no $\omega_f(y)$, such that $\omega_f(x) \subset \omega_f(y)$, $\omega_f(x) \neq \omega_f(y)$.

In their paper [10], Schweizer and Smítal have defined the notion of distributional chaos. Several authors have continued exploring this notion in various settings. The notion is based on distributional functions of pairs of points. For any $x, y \in X$, a positive integer n and real t put

$$\xi(x, y, t, n) = \#\{i; 0 \leq i \leq n-1 \text{ and } \delta_{x,y}(i) < t\} \tag{1}$$

where $\delta_{x,y}(i) = \mathrm{d}\left(f^i(x), f^i(y)\right)$ is the distance of ith iteration of the points x and y. Put

$$F^*_{x,y}(t) = \limsup_{n\to\infty} \frac{1}{n}\xi(x, y, t, n), \tag{2}$$

$$F_{x,y}(t) = \liminf_{n\to\infty} \frac{1}{n}\xi(x, y, t, n). \tag{3}$$

Function $F^*_{x,y}$ is called an *upper distributional function* and $F_{x,y}$ is called a *lower distributional function*. Both $F^*_{x,y}$ and $F_{x,y}$ are nondecreasing functions with $F^*_{x,y}(t) = F_{x,y}(t) = 0$ for every $t < 0$ and $F^*_{x,y}(t) = F_{x,y}(t) = 1$ for every $t > \operatorname{diam} X$.

An *arc* is a topological space homeomorphic to the compact interval $[0, 1]$. A *graph* G is a continuum (a nonempty compact connected metric space) which can be written as the union of finitely many arcs any two of which can intersect only in their endpoints. If a graph does not contain a set homeomorphic to the circle, it is called a *tree*. A *subgraph* is a subset of a graph which itself is a graph.

By a *periodic subgraph* we mean a subgraph $H \subset G$, such that there is an $n \geq 1$ for which $H, f(H), \dots, f^{n-1}(H)$ have pairwise disjoint interiors and $f^n(H) = H$; in this case, we also speak about a *periodic orbit* of subgraphs.

A nonempty intersection of three or more arcs, with pairwise disjoint interiors, is called a *vertex*, where the maximal number of such arcs is called the *degree of the vertex*. The *maximal order of a graph* is the maximum number of degrees of vertices on the graph.

We say that two points $u, v \in G$ are *synchronous* if both $\omega_f(u)$ and $\omega_f(v)$ are contained in the same maximal omega-limit set ω, and if, for any periodic subgraph H such that $\operatorname{Orb}_f(H) \supset \omega$, there is a $j \geq 0$ such that $f^j(u), f^j(v) \in H$. The *spectrum* of f, denoted by $\Sigma(f)$, is a set of minimal elements of the set $D(f) = \{F_{u,v};\ u \text{ and } v \text{ are synchronous}\}$. And the *weak spectrum* $\Sigma_w(f)$ of f is a set of minimal elements of the set $D_w(f) = \{F_{u,v};\ \liminf_{i\to\infty} \delta_{u,v}(i) = 0\}$.

The relation between $\Sigma(f)$ and $\Sigma_w(f)$ for the case of continuous interval self maps is discussed in *Remark* 2.6 [10], and the authors have shown that in this case there holds $\Sigma(f) \subset D_w(f)$, but we can not say anything about $\Sigma(f) \cap \Sigma_w(f)$. In the case of continuous tree self maps, like in the case of continuous interval self maps, there is no direct relation between $\Sigma(f)$ and $\Sigma_w(f)$, but the arguments from [10] can be applied to this case and, therefore, it is true that $\Sigma(f) \subset D_w(f)$.

The following theorem is our main result. It generalizes the result in [10] for the interval maps and in [5] for the circle maps to the case of tree maps.