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Ideal-convergence classes

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A B S T R A C T

Let X be a non-empty set. We consider the class $\mathcal{C}$ consisting of triads $(s, x, \mathcal{I})$, where $s = (s_d)_{d \in D}$ is a net in X, $x \in X$, and $\mathcal{I}$ is an ideal of D. We shall find several properties of $\mathcal{C}$ such that there exists a topology τ for X satisfying the following equivalence: $((s_d)_{d \in D}, x, \mathcal{I}) \in \mathcal{C}$, where $\mathcal{I}$ is a proper D-admissible ideal on D, if and only if $(s_d)_{d \in D}$ $\mathcal{I}$-converges to x relative to the topology τ.

1. Introduction

It is known that sequences fail to describe important topological properties, like closure, continuity, and compactness. These notions, which hold for example for metric spaces, are no longer true for general topological spaces. This inadequacy of sequences is due to the fact that the convergence of sequences in a space does not uniquely determine its topology (see, for example, [5,11,28]).

In [7] Franklin gave the characterization of the class of topological spaces which can be specified completely by the knowledge of their convergent sequences. He showed that this is the class of sequential spaces and proved that a space is sequential if and only if it is the quotient of some metric space.

Nets and filters are defined to overcome the shortcomings of sequences. Kelley in [10] extended and simplified results of Birkhoff [1] and Tukey [26] on Moore–Smith convergence. He gave an answer to the following problem: Given a class $\mathcal{C}$ of pairs (s, x), where s is a net in X and x is a point of X, what conditions

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on this class will guarantee the existence of a unique topology τ on X, such that $(s, x) \in \mathcal{C}$ if and only if s converges to x, with respect to the topology τ?

Subsequently the concept on convergence of a sequence has been extended to statistical convergence by Fast [6], Fridy [8], Šalát [21], Schoenberg [23], Steinhaus [24], and Zygmund [27]. This convergence has many applications in mathematical analysis and number theory (see [2–4,16,17]).

Finally, the concept of $\mathcal{I}$-convergence of a sequence of real numbers, where $\mathcal{I}$ in an ideal of natural numbers, was first introduced in [12] and extended to general spaces in [13,14,19,22]. The $\mathcal{I}$-convergence is a natural generalization of usual convergence and statistical convergence of sequences. In particular, if $\mathcal{I}$ is the ideal of finite subsets of natural numbers, we have the usual convergence and if $\mathcal{I}$ is the ideal of subsets of natural numbers with zero density, we have the statistical convergence (see, for example, [9,18,20,25]). In [15] Lahiri and Das introduced and investigated the idea of $\mathcal{I}$-convergence to nets in general topological spaces.

In this paper we introduce and study the notion of ideal-convergence class. Especially, in Section 2 we give some preliminaries which will be used later. In Section 3 we present basic propositions necessary for the proof of the main theorem which is proved in Section 4:

Theorem. *Let $\mathcal{C}$ be an ideal-convergence class for a set X. We consider the function* $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$, *where* $\mathrm{cl}(A)$ *is the set of all $x \in X$ such that, for some net $(s_d)_{d\in D}$ in A and a proper ideal $\mathcal{I}$ of the directed set D, $(s_d)_{d\in D}$ $\mathcal{I}$-converges $(\mathcal{C})$ to x. Then,* cl *is a closure operator on X and $((s_d)_{d\in D}, x, \mathcal{I}) \in \mathcal{C}$, where $\mathcal{I}$ is a proper D-admissible ideal, if and only if $(s_d)_{d\in D}$ $\mathcal{I}$-converges to x relative to the topology associated with* cl.

2. Preliminaries

In this section, we recall some of the basic concepts related to the convergence of nets in topological spaces and we refer [11] for more details.

Let D be a non-empty set. A non-empty family $\mathcal{I}$ of subsets of D is called *ideal* if $\mathcal{I}$ has the following properties:

(1) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(2) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

The ideal $\mathcal{I}$ is called *proper* if $D \notin \mathcal{I}$.

A partially ordered set D is called *directed* if every two elements of D have an upper bound in D.

If $(D, \leqslant_D)$ and $(E, \leqslant_E)$ are directed sets, then the Cartesian product $D \times E$ is directed by $\leqslant$, where $(d_1, e_1) \leqslant (d_2, e_2)$ if and only if $d_1 \leqslant_D d_2$ and $e_1 \leqslant_E e_2$. Also, if $(E_d, \leqslant_d)$ is a directed set for each d in a set D, then the product

$$\prod_{d\in D} E_d = \{f : D \to \bigcup_{d\in D} E_d : f(d) \in E_d \text{ for all } d \in D\}$$

is directed by $\leqslant$, where $f \leqslant g$ if and only if $f(d) \leqslant_d g(d)$, for all $d \in D$.

A net in a set X is an arbitrary function s from a non-empty directed set D to X. If $s(d) = s_d$, for all $d \in D$, then the net s will be denoted by the symbol $(s_d)_{d\in D}$.

A net $(t_\lambda)_{\lambda\in\Lambda}$ in X is said to be a *semisubnet* of the net $(s_d)_{d\in D}$ in X if there exists a function $\varphi : \Lambda \to D$ such that $t = s \circ \varphi$. We write $(t_\lambda)^{\varphi}_{\lambda\in\Lambda}$ to indicate the fact that φ is the function mentioned above.

A net $(t_\lambda)_{\lambda\in\Lambda}$ in X is said to be a *subnet* of the net $(s_d)_{d\in D}$ in X if there exists a function $\varphi : \Lambda \to D$ with the following properties: